ECE 359 **Spring 2003**

Handout # 2 January 23, 2003

MATH REVIEW

COMPLEX VARIABLES

A complex number z is expressed as

$$z = x + jy,$$

where x and y are real numbers, and j is the square root of -1 (i.e., $j^2 = -1$ and $1/j = -j$)

The *real part* of z is denoted by $\mathrm{Re}(z)$ and the *imaginary part* by $\mathrm{Im}(z)$. That is,

$$x = \mathrm{Re}(z) \quad \text{and} \quad y = \mathrm{Im}(z)$$

The *complex conjugate* of z is denoted by z^*, and $z^* = x - jy$. Clearly,

$$\mathrm{Re}(z) = \frac{z + z^*}{2} \qquad \mathrm{Im}(z) = \frac{z - z^*}{2j}$$

The *magnitude* of z is denoted by $|z|$, and the *angle* (or phase) of z is denoted by $\angle z$.

$$|z| = \sqrt{x^2 + y^2} = \sqrt{[\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2} \qquad \angle z = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right)$$

With this understanding, z can be expressed in terms of $|z|$ and $\angle z$ in the following way.

$$z = |z| \left(\cos(\angle z) + j\sin(\angle z)\right) \tag{1}$$

Exercise: Prove that Equation (1) is true.

Example: $z = 4 + 7j$

$$\mathrm{Re}(z) = 4 \quad \mathrm{Im}(z) = 7$$

$$|z| = \sqrt{4^2 + 7^2} = 8.062 \qquad \angle z = \tan^{-1}(7/4) = 1.05 \text{ radians}$$

Complex Exponential Function: For any real number a, the *complex exponential function* of a is defined by

$$e^{ja} = \cos a + j\sin a$$

Plugging the above in Equation (1), we get

$$z = |z|\, e^{j\angle z}$$

This is called the *polar form* of the complex number z.

The complex exponential function has properties similar to the real exponential function that you are very familiar with. For example,

$$e^{j(a+b)} = e^{ja}\, e^{jb} \qquad e^{j(a-b)} = \frac{e^{ja}}{e^{jb}} \qquad \int e^{ja} da = \frac{e^{ja}}{j} \qquad \frac{d}{da} e^{ja} = je^{ja}$$

All other operations using the complex exponential function such as integration by parts, chain rule, etc. work exactly like those on the real exponential function with the understanding that we need to keep track of the "j" in the expressions.

Operations on Complex Numbers

Let z_1 and z_2 be any two complex numbers. Then

Addition and Subtraction:

$$z_1 + z_2 = \mathrm{Re}(z_1) + \mathrm{Re}(z_2) + j[\mathrm{Im}(z_1) + \mathrm{Im}(z_2)]$$

$$z_1 - z_2 = \mathrm{Re}(z_1) - \mathrm{Re}(z_2) + j[\mathrm{Im}(z_1) - \mathrm{Im}(z_2)]$$

Multiplication and Division:

Examples:

$$(2+3j)(4+j) = 8 + 2j + 12j + 3j^2 = 5 + 14j \quad \text{(remember } j^2 = -1\text{)}$$

$$\frac{2+3j}{4+j} = \frac{(2+3j)(4-j)}{(4+j)(4-j)} = \frac{11+10j}{17}$$

Multiplication and Division are easier in polar form:

$$z_1 z_2 = |z_1|\,|z_2|\, e^{j(\angle z_1 + \angle z_2)}$$

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} e^{j(\angle z_1 - \angle z_2)}$$

Exponentiation: (also easier in polar form)

$$(1+j\sqrt{3})^4 = \left(2e^{j\pi/3}\right)^4 = 2^4 e^{j4\pi/3} = 16(\cos(4\pi/3) + j\sin(4\pi/3)) = -8 - 8\sqrt{3}j$$

Roots: (also easier in polar form)

Example: Find all the cube roots of 8

$$8^{1/3} = \left(8e^{j0}\right)^{1/3} = 8^{1/3} e^{j(0+2\pi k/3)}, k = 0,1,2 \;=\; 2, 2e^{j2\pi/3}, 2e^{j4\pi/3}$$

<u>TRIGONOMETRIC IDENTITIES</u>

$$\cos(-x) = \cos(x) \qquad \sin(-x) = -\sin(x) \qquad \cos^2(x) + \sin^2(x) = 1$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \qquad \sin(x+y) = \sin x \cos y + \sin y \cos x$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x \qquad \sin(2x) = 2\sin x \cos x$$

$$\cos^2 x = \frac{1+\cos(2x)}{2} \qquad \sin^2 x = \frac{1-\cos(2x)}{2}$$

$$\cos x \cos y = \frac{\cos(x+y)+\cos(x-y)}{2} \qquad \sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x \sin y = \frac{\cos(x-y)-\cos(x+y)}{2} \qquad \sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\sin x \cos y = \frac{\sin(x+y)+\sin(x-y)}{2} \qquad \cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin y \cos x = \frac{\sin(x+y)-\sin(x-y)}{2} \qquad \cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos x = \frac{e^{jx}+e^{-jx}}{2} \qquad \sin x = \frac{e^{jx}-e^{-jx}}{2j}$$

$$A\cos(x) + B\sin(x) = \sqrt{A^2+B^2}\,\cos(x - \tan^{-1}(B/A))$$

LOGARITHMS

Definition: The log function is the inverse of the exponential function.

$$a^x = b \;\Leftrightarrow\; \log_a b = x$$

where a is the base of the log function. Commonly used bases are 10 and e (for natural log or the ln function). If a base is not specified it is usually taken to be 10.

Properties:

End point conditions: $\log_a a = 1 \quad \log_a(0) = -\infty \ (a > 1) \quad \log_a(1) = 0 \quad \log_a(\text{negative num}) = \text{undefined}$

Base conversion: $\log_a b = \dfrac{\ln(b)}{\ln(a)} = \dfrac{\log(b)}{\log(a)} = \dfrac{\log_c b}{\log_c a}$

Multiplication and Division: $\log_a(xy) = \log_a x + \log_a y \qquad \log_a(x/y) = \log_a x - \log_a y$

Exponents in logs: $\log_a(b^r) = r\log_a b$

Inverse operations: $a^{\log_a x} = x \qquad \log_a a^x = x \qquad 2^x = e^{\ln(2^x)} = e^{x\ln 2}$, etc.

SEQUENCES AND SERIES

Taylor Series: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, where $f^{(n)}(a)$ is the n-th derivative of f evaluated at a.

Examples

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \quad \text{for } |x| < 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Other Infinite Series

Euler's Definition of e: $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$

Finite Geometric Series: $a\sum_{i=0}^{n} r^i = a + ar + ar^2 + \cdots + ar^n = \frac{a(1-r^{n+1})}{1-r}$

Infinite Geometric Series: $a\sum_{i=0}^{\infty} r^i = a + ar + ar^2 + \cdots = \frac{a}{1-r}$, if $|r| < 1$

Sum of Numbers: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Sum of Squares: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$