Rate-Based versus Queue-Based Models of Congestion Control

Supratim Deb* and R. Srikant†

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Abstract

Mathematical models of congestion control capture the congestion indication mechanism at the router in two different ways: rate-based models, where the queue-length at the router does not explicitly appear in the model, and queue-based models, where the queue length at the router is explicitly a part of the model. Even though most congestion indication mechanisms use the queue length to compute the packet marking or dropping probability to indicate congestion, we argue that, depending upon the choice of the parameters of the AQM scheme, one would obtain a rate-based model or a rate-and-queue-based model as the deterministic limit of a stochastic system with a large number of users.

1 Introduction

Deterministic fluid-flow models have been widely used [25, 23, 20, 4] to describe congestion control and active queue management (AQM) schemes in the Internet. These models capture the mean behavior of the congestion controlled sources. All of these models use a packet marking (or packet dropping) function to describe the fraction of packets marked (or dropped) at a link. Depending upon the model, the marking function is either a function of the queue length [18, 16] or a function of the instantaneous arrival rate at the router [19, 23]. In this paper, we consider AQM schemes where the router decides the fraction of packets to be marked based on the occupancy level of a real or virtual queue [15]. We study appropriate models for the deterministic marking function based on the parameters of the virtual queue based AQM scheme.

Judiciously designed AQM schemes can lead to small queue lengths at the routers by providing early warning about incipient congestion to the sources accessing the network. Such mechanisms include RED [14],

*Supratim Deb is with the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology. e-mail: supratim@mit.edu

†R. Srikant is with the Department of Electrical and Computer Engineering and Coordinated Science Laboratory, University of Illinois at Urbana-Champaign. e-mail:rsrikant@uiuc.edu

REM [3], and PI [18]. All of these mechanisms compute a marking or dropping probability based on some function of the queue-length at the router. In [15], the authors propose a marking scheme based on a virtual queue which operates at a capacity slightly smaller than that of the real queue and adds a packet whenever there is an arrival into the real-queue. An Adaptive Virtual-Queue (AVQ) mechanism to drive the utilization towards a given value was proposed and analyzed in [22, 21]. It has been shown that the mean behavior of congestion controlled sources can be described by deterministic fluid-flow model [30, 10, 31], which can then be used to design the AQM parameters. However, the models in [30, 10] start by assuming that the marking probability is a function of the arrival rate at the link, or assume that the parameters of the AQM scheme are chosen such that the deterministic model explicitly includes the queue length [30, 31]. In this paper, we derive deterministic models which may or may not contain the queue length in the limit, depending upon choice of the AQM parameters.

We start with a stochastic model of a single link accessed by many congestion-controlled flows. Randomness in the congestion-controlled Internet may be due to many reasons:

- unresponsive flows which do not respond to congestion indication,
- the probabilistic nature of packet marking by an AQM scheme,
- asynchronous updates among sources,
- the inability to precisely model window flow control mechanism, and
- the initial ramp-up phase (for example, *slow start* in TCP flow control) of the congestion control mechanism.

In addition to deriving a deterministic model from this stochastic system under a limiting regime where the number of sources is large, we also derive a stochastic model to capture the deviations from the deterministic limit. We use the stochastic model to further study the performance of rate-based and queue-based models of AQM schemes.

When AQM schemes are not used, the models in [2, 27, 6] capture the randomness in the Internet, by assuming that these disturbances are independent of the arrival rate from the TCP sources. In [31], starting with a discrete-time stochastic model of a single link implementing RED which is accessed by many TCP sources, the authors have derived limiting deterministic and stochastic models of the behavior of TCP flows. With a real-queue based marking, the queue-length was shown to be $O(N)$, where N is the number of flows in the system and the capacity of the link is Nc.

For the purposes of this paper, we primarily study REM as the AQM scheme at the router. As we will comment later, the results should be applicable to a large class of queue-length based AQM schemes such as RED and its variants.

Our main contributions are as follows.

1. Depending upon the manner in which a parameter in REM is scaled with the number of flows, we argue that the limiting deterministic/stochastic model of the congestion-controlled link would capture the AQM behavior using either a rate-based or a jointly rate-and-queue-based marking function. The choice of the appropriate model for the marking function is critical in designing the parameters of the congestion control/AQM scheme.

2. We show that a virtual-queue based AQM scheme is very robust to the choice of parameter in terms of attaining a low-loss, low-delay and high-utilization operation. On the other hand, with real-queue based marking, the choice of parameter is critical if we want the queuing delay to be negligible when compared to the propagation delay. The analytical models are also corroborated through simulations.

The rest of the paper is structured as follows. The model and the problem statement is provided in Section 2. In Section 3, we discuss two different regimes of AQM parameters and obtain the appropriate models for each of these regimes. In Section 4 we discuss the equilibrium and the stability properties of a proportionally fair controller in the two different regimes. In Section 5, we precisely show the assumption which leads to deterministic models of TCP widely used in the literature and show the equilibrium values of the mean rate. In Section 6, we validate our design and analysis with simulation results. Finally, we provide concluding remarks in Section 7.

2 Basic Framework and Problem Statement

Our model is that of a single bottleneck link being accessed by many congestion controlled flows [10]. We consider discrete-time rate update model for the flows to model *TCP* and *proportionally fair* congestion controllers. The rate update mechanisms are described later in this section. The delay in the forward and the reverse path is $r/2$ sec so that the round-trip propagation delay is r sec. The number of flows in the system is N, which is also the scaling parameter. We consider a sequence of such systems indexed by N, where in the N^{th} system, there are N congestion controlled flows accessing the link. Further, in the N^{th} system there are N uncontrolled flows accessing the link. The link capacity is scaled as Nc packets per second. In this section, we consider AQM schemes which mark packets based on the queue length of a virtual queue. We denote the capacity of the virtual queue in the N^{th} system by $N\tilde{c} = N\theta c$, where $0 < \theta < 1$. We consider a slotted system, where the length of a time-slot is the time needed to serve one packet in the virtual queue which is $1/(N\tilde{c})$ s. Further, we let $\hat{q}^{(N)}[k]$ denote the length of the virtual queue at the k^{th} slot in the N^{th} system. The evolution of the virtual-queue can be described by the following.

$$\hat{q}^{(N)}[k+1] = (\hat{q}^{(N)}[k] - 1)^+ + A_c^{(N)}[k] + \sum_{i=1}^{N}(e_i^{(N)}(\tfrac{1}{N\tilde{c}})) \tag{1}$$

In the above, $A_c^{(N)}[k]$ denotes the number of arrivals due to the controlled flows in a slot and $e_i^{(N)}(t)$ is a stationary stochastic process denoting the number of arrivals due to an uncontrolled flow in a time interval of length t sec in the N^{th} system. We also denote by $e^{(N)}(t)$, the average number of arrivals due to the uncontrolled flows in time t sec so that

$$e^{(N)}(t) = \frac{1}{N}\sum_{i=1}^{N} e_i^{(N)}(t) \ .$$

2.1 Rate update mechanism

In the following, we assume that all the rates are measured in packets per second. Let $y_i^{(N)}[k]$ denote the flow rate of the i^{th} controlled flow at time slot k in the N^{th} system. Further, let $x^{(N)}[k]$ denote the average flow rate of the controlled flows through the link at time k, and so

$$x^{(N)}[k] = \frac{1}{N}\sum_{i=1}^{N} y_i^{(N)}[k] \ .$$

In a window based implementation, the rate update interval of the congestion-controlled flows is taken to be the time between two successive ACKs, which is roughly equal to the inverse of the equilibrium rate seen by the controlled flow. Since $N\tilde{c}$ is the capacity of the virtual queue, the rate update interval will be approximately equal to $1/\tilde{c}$. Thus $y_i^{(N)}[k]$ is updated once every N slots in the N^{th} system.

Denote by $\tilde{x}_l^{(N)}$ the following quantity.

$$\tilde{x}_l^{(N)} = \frac{1}{N}\sum_{i=1}^{N} \tilde{y}_l^{(N)} \ .$$

Before we introduce the congestion control model with randomness, we first introduce the deterministic models of congestion control used in the literature. This will help us understand the rationale behind our stochastic model when we describe it later. Suppose, in the interval $[Nk, N(k+1)]$, the link marks $p_{det}[Nk]$ fraction of the packets. Then, the deterministic rate update model with a proportionally fair controller is given by

$$y_i^{(N)}[N(k+1)] - y_i^{(N)}[Nk] = \kappa[w - y_i^{(N)}[N(k-d+1)]p_{det}[N(k+1-\frac{d}{2})]] \ . \tag{2}$$

In reality, the quantity $y_i^{(N)}[N(k-d+1)]p_{det}[N(k+1-\frac{d}{2})]$, which denotes the rate at which marked packets are received, is not deterministic. In fact, $y_i^{(N)}[N(k-d+1)]p_{det}[N(k+1-\frac{d}{2})]$ should be replaced by number of marked packets received (which is a random variable) in the rate update interval divided by the length of the rate update interval.

Let $M_i^{(N)}[k]$ denote the number of packets of source i marked by the link at time k. Let τ denote the rate update interval of the controlled flows. Since the slot length in the N^{th} system is $1/(N\tilde{c})$, the round trip propagation delay of the controlled flows in the N^{th} system is clearly Nd slots, where $d = r/\tau$. Note that d can also be viewed as the number of updates of source i in a round trip time. For simplicity we will assume that d is an even integer.

We describe two popular congestion controllers in the following.

• **Proportionally fair controller:** With proportionally fair congestion controller, the update of the i^{th} source is described by

$$y_i^{(N)}[N(k+1)] - y_i^{(N)}[Nk] = \kappa(w - \frac{1}{\tau}\sum_{s=N(k-\frac{d}{2})+1}^{N(k+1-\frac{d}{2})} M_i^{(N)}[s])\ , k = 0, 1, 2 \ldots . \tag{3}$$

$$y_i^{(N)}[Nk] = y_i^{(N)}[Nk+l] \text{ for } l \in \{0, 1, 2, \ldots N-1\} \tag{4}$$

Since there are d updates in a round trip time, the marks received between the k^{th} and the $(k+1)^{th}$ update is a fraction of packets sent after the $(k-d+1)^{th}$ update (and before the $(k-d+2)^{th}$ update) which are $\tau y_i^{(N)}[N(k-d+1)]$ in number. Thus, if the link decides to mark $p_{det}[N(k-d/2+1)]$ fraction of these packets in the time interval $[N(k-\frac{d}{2})+1, N(k+1-\frac{d}{2})]$ and the sources have a perfect estimate of $p_{det}[N(k-d+1)]$, then

$$\sum_{s=N(k-\frac{d}{2})+1}^{N(k+1-\frac{d}{2})} M_i^{(N)}[s] = \tau p_{det}[N(k-\frac{d}{2}+1)]y_i^{(N)}[N(k+1-d)]$$

and we get the familiar form of a *proportionally fair* controller. However, $M_i^{(N)}[s]$ is random and is determined by the marking mechanism at the router, and we will discuss a model for this in the next section. We note that, if the updated rates become negative, the updated rates should be projected into the positive axis. However, for the purposes of the derivations in this paper, we do not show this explicitly in the rate update of the individual flows. It was shown in [30] for a continuous time version of the system with unresponsive flows but without probabilistic marking that, in a many flows regime, the average flow rate behaves as though the non-negativity constraint is ignored. Thus, in deriving the behavior of the average flow rate, we ignore the non-negativity constraint.

• **TCP-like or minimum potential delay controller:** We use the following model for the rate update of TCP:

$$y_i^{(N)}[N(k+1)] - y_i^{(N)}[Nk] = \kappa(w - \frac{1}{\tau} y_i^{(N)}[Nk] \sum_{s=N(k-\frac{d}{2})+1}^{N(k+1-\frac{d}{2})} M_i^{(N)}[s]) \ . \tag{5}$$

The standard TCP is recovered with $w = 1/r^2$. Again, if the link decides to mark $p_{det}[N(k - d/2 + 1)]$ fraction of these packets in the time interval $[N(k - \frac{d}{2}) + 1, N(k + 1 - \frac{d}{2})]$ and the sources have a perfect estimate of $p_{det}[N(k - d + 1)]$, then

$$\sum_{s=N(k-\frac{d}{2})+1}^{N(k+1-\frac{d}{2})} M_i^{(N)}[s] = \tau p_{det}[N(k-d+1)] y_i^{(N)}[N(k+1-d)] \ ,$$

and we get the rate update model for the congestion avoidance phase of TCP, variants of which have been proposed in [23, 31, 28, 7].

2.2 Queue-based marking

In a queue-based marking mechanism, packets are marked based on the queue length [14, 3, 18, 1, 33, 24]. The queue length could be the length of the real queue or the length of a virtual queue. The notion of a virtual queue was first introduced in [15]. A packet is added to the virtual queue whenever there is a packet arrival into the real queue, but packets are drained from the virtual queue at a rate $\tilde{C} = N\tilde{c} = N\theta c$ ($\theta \leq 1$). If $\theta < 1$, the capacity of the virtual queue is smaller than the capacity of the real queue. Thus, marking based on the queue length at the virtual queue helps to detect incipient congestion. Throughout the paper, we use $\hat{q}$ to denote the length of virtual-queue. We also use $f(\hat{q})$ to denote the marking profile at the link which depends on the length of the virtual queue $\hat{q}$.

As mentioned before, we will assume that marking is based on the length of the virtual queue. We can recover the case of marking based on the real queue by letting $\theta = 1$. In the case of virtual-queue-based marking, the RTT can be well-approximated by the propagation delay since the queuing delay will be negligible. However, in the case of real-queue-based marking, the RTT is the sum of the propagation delay and the queuing delay, which is variable. Nevertheless, for our modeling purposes, in that case, we still approximate the RTT by a constant (equal to the sum of the propagation delay and the equilibrium queuing delay).

In a virtual-queue based REM, a packet is marked with probability

$$f(\hat{q}) = 1 - \exp(-\gamma^{(N)} \hat{q})$$

when the length of the virtual queue is $\hat{q}$. The critical parameter in REM is $\gamma^{(N)}$. We consider two regimes of parameters to study the robustness of a virtual-queue mechanism to the choice of parameters. In the context of REM, we consider $\gamma^{(N)} = \gamma/N$ and $\gamma^{(N)} = \gamma$.

In addition to REM, the analysis can be applied to other AQM schemes like RED [13]. The marking profile for RED is given by

$$f(\hat{q}) = \min(1,\ \hat{\alpha}(\hat{q} - \hat{q}_{min})^+)$$

where, $\hat{\alpha}$ and $\hat{q}_{min}$ are parameters of the marking function. Thus, our analysis can be applied to RED with the choice of $\hat{\alpha}, \hat{q}_{min}$ as $\hat{\alpha} = \alpha/N, \hat{q}_{min} = Nq_{min}$ and, $\hat{\alpha} = \alpha, \hat{q}_{min} = q_{min}$.

2.3 Model for packet arrivals at the router

The queuing phenomenon at the router happens at a faster time scale compared to the rate update mechanism at the sources. As mentioned earlier, the sources' rate updates occur approximately once every $1/\tilde{c}$ sec., whereas the time to process a packet at the router is $1/Nc$ sec. Due to the features of the window flow control mechanism that we have not modeled, there can be a lot variability in the number of packets received at the router at time scales of the order of $1/Nc$. For example, the window implementation of flow control implies that the rate updates of the sources are not in regular intervals (we use $\tau = 1/\tilde{c}$ as roughly the mean update interval at the sources). Further, the sources are not synchronized; thus, the packets from a source can arrive at the link at any point of time within a slot. We make the following assumption to capture the variability of the arrival rate into the router due to these additional factors not reflected in the discrete-time rate update model.

Assumption 1. *(Arrival process at the link)*
If the transmission rate of the i^{th} controlled flow is x_i, the number of packets arriving at the link over a time interval of length $1/(N\tilde{c})$ is Poisson distributed with mean $x_i/(N\tilde{c})$. □

Our simulation results later in the chapter support this assumption.

2.4 Problem statement

We attempt to answer the following question in this chapter.

- With REM as a marking mechanism, given a choice of $\gamma^{(N)}$ which can be either $\gamma^{(N)} = \gamma/N$ or $\gamma^{(N)} = \gamma$, what is the suitable model for the marking function to predict the steady state properties like mean and variance of the arrival process into the link?

We will also provide design rules to guarantee low-loss and low queuing-delay at the link, along with high utilization. Specifically, we are interested in developing models to provide guidelines to answer the following questions:

- Suppose the system capacity is fixed. How should we choose the parameters of the marking function so that the probability of buffer overflow is smaller than a given target?
- Suppose the parameters of the marking function are fixed. What should be the link capacity so that the probability of buffer overflow is smaller than a given target?

In the rest of the paper, we use the abbreviation $a.s.$ for "almost surely," and also use the notation $X_n \stackrel{d}{\rightarrow} X$ to imply that the sequence of random variables X_n's converges in distribution to X (which is either a random variable or a deterministic quantity).

3 Queue-Based Marking and Scaling Regimes

We recall that REM [3] marks a packet with probability $(1 - \exp(-\gamma^{(N)}\hat{q}))$, if there are $\hat{q}$ packets in the virtual queue. The crucial parameter in REM mechanism is $\gamma^{(N)}$. Prescriptions for the values of $\gamma^{(N)}$ are provided in [5] based on extensive simulations. Roughly speaking, a larger value of $\gamma^{(N)}$ leads to faster convergence of the rate-control mechanism, but at the cost of smaller utilization at the link. To this end, as mentioned earlier, we consider two choices for the parameter $\gamma^{(N)}$: $\gamma^{(N)} = \gamma/N$ and $\gamma^{(N)} = \gamma$.

For the purposes of the results in this section, we make the following assumption regarding the arrival process due to uncontrolled flows.

Assumption 2. *In the N^{th} system, the number of arrivals from the uncontrolled flows in a slot (which is of length $1/(N\tilde{c})$ is Poisson($a/\tilde{c}$).*

This assumption, which can be relaxed, primarily helps us to derive closed form expressions for the suitable marking functions as we will see later. We point out that this is reasonable in the following sense. It is well-known that, if there are N identical stationary point processes, then the process obtained by superposing the N processes and dilating the time scale by a factor of N goes to a Poisson process in the limit of large N [9]. Thus, the assumption that the number of packets from the N controlled flows over a time-interval of length $1/(N\tilde{c})$ is Poisson is reasonable.

We next describe the stochastic models for the proportionally fair controller and the TCP-like controller in the rest of this section.

3.1 REM with $\gamma^{(N)} = \gamma/N$

Let $\overline{q}^{(N)}[k] = \hat{q}^{(N)}[k]/N$ for the purpose of this subsection. In this parameter regime, the marking probability can be expressed as a function of $\overline{q}^{(N)}$ as follows:

$$p(\overline{q}^{(N)}) = 1 - \exp(-\gamma \overline{q}^{(N)}) \ . \tag{6}$$

We also call $\overline{q}^{(N)}[k]$ the normalized virtual queue-length. Also let $\tilde{q}_l^{(N)} = \overline{q}^{(N)}[Nl]$ for $l = 1, 2, 3 \ldots$.

3.1.1 Proportionally fair controller

We first derive a model for the limiting marking function that will help us describe the limiting rate-update behaviors. The number of marks received by a source between two updates in this parameter regime depends on $\overline{q}^{(N)}[k]$ over a rate update interval. We show that the number of marks received by a source in this regime can be expressed as a function of $\tilde{q}_l^{(N)}$ (i.e., the normalized virtual lengths at instants $Nk, k = 1, 2, \ldots$) and the average flow rate of the controlled flows. We state the result precisely in the form of following theorem. We remind the reader that, τ denotes the length of the rate update interval of a controlled flow and is equal to $1/\tilde{c}$ in our model.

Theorem 3.1. *Consider a time interval of length τ over which the rates of the controlled flows do not change. Further, let $y_i^{(N)}$, $x^{(N)}$ and, $\overline{q}^{(N)}$ be the rate of the i^{th} controlled flow into the link, the average rate of the N controlled flows into the link, and the normalized virtual queue-length respectively at the beginning of the time interval under consideration. Also let $M_i^{(N)}$ be the random variable denoting the number of packets of the i^{th} controlled flow marked in this interval of length τ. If $y_i^{(N)} \stackrel{d}{\to} y_i$, $x^{(N)} \stackrel{a.s.}{\to} x$ and $\overline{q}^{(N)} \stackrel{a.s.}{\to} q$ as $N \to \infty$, then under Assumptions 1 and 2, we have in the parameter regime $\gamma^{(N)} = \gamma/N$,*

$$\lim_{N \to \infty} M_i^{(N)} \stackrel{d}{=} \textit{Poisson}(\tau y_i p_q(q, x)) \tag{7}$$

where

$$p_q(q, x) = 1 - \exp(-\gamma q) \frac{\exp\left(\gamma \min(\, q, 1 - \frac{x+a}{\tilde{c}}\,)\right) - 1}{\gamma\left(1 - \frac{x+a}{\tilde{c}}\right)} - \left(1 - \frac{\min\left(q, 1 - \frac{x+a}{\tilde{c}}\right)}{1 - \frac{x+a}{\tilde{c}}}\right) \ . \tag{8}$$

Proof. Note that the interval of length τ sec. has N slots. Let those slots be indexed by $m, m+1, m+2, \ldots m+N-1$. Recall that, by Assumption 1, the number of arrivals from the i^{th} controlled flow in a slot is Poisson($y_i^{(N)}/(N\tilde{c})$).

In the N^{th} system, the virtual queue length process evolves as discrete-time queue with service rate one per slot and the number of arrivals in a slot distributed as Poisson($(x^{(N)} + a)/\tilde{c}$). Let $M_i^{(N)}[k]$ denote the number of packets of source i marked in slot k ($k = m, m+1, m+2, \ldots m+N-1$). If the packets are marked upon arrival at the virtual-queue, then $M_i^{(N)}[k]$ is distributed as Poisson($p(\overline{q}^{(N)}[k])y_i^{(N)}/(Nc)$) which follows from the fact that a probabilistic splitting of a Poisson random variable gives a Poisson random variable. Here $p(\overline{q}^{(N)}[k])$ is the marking probability which in this parameter regime is given by (6). We are interested in finding the distribution of $M_i^{(N)} = \sum_{k=m}^{m+N-1} M_i^{(N)}[k]$. Define $q^{(N)}(t)$ as follows:

$$q^{(N)}(t) = \overline{q}^{(N)}[\lfloor N\tilde{c}t \rfloor + m] = \frac{\hat{q}^{(N)}[\lfloor N\tilde{c}t \rfloor + m]}{N} \ , \tag{9}$$

where $t \in [0, 1/\tilde{c}]$. It is well known that [8], in the limit of large N, the process $q^{(N)}(t)$ behaves like a deterministic process (in the interval under consideration). Let $A_c^{(N)}[k]$ denote the number of arrivals due to the controlled flows in slot k. Since $A_c^{(N)}[k] \sim$ Poisson($x^{(N)}/\tilde{c}$) and also $x^{(N)} \stackrel{a.s.}{\to} x$, we have from a triangular array version of strong law of large numbers that [12]

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1} \lfloor N\tilde{c}t \rfloor A_c^{(N)}[k] \stackrel{a.s.}{=} xt$$

It follows [8] that,

$$\lim_{N\to\infty} q^{(N)}(t) = q(t) \text{ u.o.c.}, \tag{10}$$

$$q(t) = (q + t(x + a - \tilde{c}))^+ \ . \tag{11}$$

In the above, the convergence is in an almost sure sense and is uniformly over compact sets.

Next note that given $\{\hat{q}^{(N)}[k], k = m, m+1, m+2 \ldots m+N-1\}, y_i^{(N)}$

$$\begin{aligned} M_i^{(N)} &\sim \text{Poisson}(\tfrac{y_i^{(N)}}{N\tilde{c}} \textstyle\sum_{k=m}^{m+N-1} p(\overline{q}^{(N)}[k])) \\ &= \text{Poisson}(y_i^{(N)} \tau \tfrac{1}{N} \textstyle\sum_{k=0}^{N-1} p(\overline{q}^{(N)}[k+m])) \end{aligned} \tag{12}$$

Further, observe the following.

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} p(\overline{q}^{(N)}[k+m]) \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} p(q^{(N)}\left(\frac{k}{N\tilde{c}}\right)) \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} p(q\left(\frac{k}{N\tilde{c}}\right)) \ + \ \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} [p(q^{(N)}\left(\frac{k}{N}\right)) - p(q\left(\frac{k}{N}\right))] \ . \end{aligned}$$

The second term in the preceding expression goes to zero, since $q^{(N)}(t)$ converges to $q(t)$ uniformly over the interval $[0, \tau]$ and further since the marking profile, $p(\cdot)$ is bounded from above by 1. The first term in the expression converges to an appropriate integral and so we have the following in the limit of large N:

$$
\begin{aligned}
&\tau \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} p(\overline{q}^{(N)}[k+m]) \\
&\overset{a.s.}{=} \quad \int_0^{1/\tilde{c}} p(q(s))ds \\
&\overset{a.s.}{=} \quad \int_0^{1/\tilde{c}} p((q_0 + s(x+a-\tilde{c}))^+)ds \\
&\overset{a.s.}{=} \quad \tau \left(1 - \exp(-\gamma q_0) \frac{\exp\left(\gamma \min\left(q_0, 1 - \frac{x+a}{\tilde{c}}\right)\right) - 1}{\gamma\left(1 - \frac{x+a}{\tilde{c}}\right)} - \left(1 - \frac{\min\left(q_0, 1 - \frac{x+a}{\tilde{c}}\right)}{1 - \frac{x+a}{\tilde{c}}}\right)\right) .
\end{aligned}
$$

Further since $y_i^{(N)} \xrightarrow{d} y_i$, it is easy to see that

$$y_i^{(N)} \tau \tfrac{1}{N} \textstyle\sum_{k=0}^{N-1} p(\overline{q}^{(N)}[k+m]) \xrightarrow{d} y_i \tau p_q(q_0, x)$$

The result thus follows from (12). □

Remark 1. *1. The result given by Theorem 3.4 gives an expression for the mean number of marked packets of the i^{th} source. Since τy_i is the mean number of packets sent by source i in the rate update interval, $p_q(q, x)$ as given in (8) can be interpreted as the equivalent marking function in this parameter regime, which a function of the normalized virtual queue-length at instants Nk ($k = 1, 2, \ldots$) and the average rate x.*

2. In the typical operating regime, γ is a small number and the average rate of the controlled flows is close to the target equilibrium value of $\tilde{c} - a$ and so the equivalent marking function can be approximated by

$$p_q(q, x) \approx 1 - \exp(-\gamma q) \; . \tag{13}$$

Under this approximation, the marking function depends only on the normalized queue length at the boundaries of the rate update interval. This modeling assumption has been widely used in the literature

□

We are now in a position to describe the limiting behavior of the rate-update using the equivalent marking function just derived. We have following result describing the behavior of proportionally fair controller. The proof uses induction in time as in [31] and the equivalent marking function derived in Theorem 3.4.

Theorem 3.2. *For a proportionally-fair controller and REM with $\gamma^{(N)} = \gamma/N$, as $N \to \infty$, $x^{(N)}[Nl] \stackrel{a.s.}{\to} \tilde{x}_l$, $\overline{q}^{(N)}[Nl] \stackrel{a.s.}{\to} \tilde{q}_l$, and $y_i^{(N)}[Nl] \stackrel{d}{\to} \tilde{y}_l$, where*

$$\tilde{y}_{(l+1)} - \tilde{y}_l = \kappa \left[w - \frac{1}{\tau} M_{(l+1-\frac{d}{2})} \right] \tag{14}$$

$$M_l \sim \textit{Poisson}(\tau \tilde{y}_{(l-\frac{d}{2})} p_q(\tilde{q}_l, \tilde{x}_{(l-\frac{d}{2})})) \tag{15}$$

$$\tilde{x}_{(l+1)} - \tilde{x}_l = \kappa \left[w - \tilde{x}_{(l+1-d)} p_q(\tilde{q}_{(l+1-\frac{d}{2})}, \tilde{x}_{(l+1-d)}) \right] \tag{16}$$

$$\tilde{q}_{(l+1)} = (\tilde{q}_l + \tau(x_{(l-\frac{d}{2})} + a - \tilde{c}))^+ \ . \tag{17}$$

Proof. We use induction in time as in [31]. For simplicity, we will assume that the initial conditions are identical for all the users in the time interval $[-r, 0]$ seconds, and the queue starts from zero initially. The result is then trivially true for $l \in [0, d]$ (recall that d is the number of updates in a round trip time of r seconds). We will assume that the result is true for l (i.e., after the l^{th} update) and prove the result for $(l+1)$.

Denote by $q_l^{(N)}(t)$ the following.

$$q_l^{(N)}(t) = \overline{q}^{(N)}[Nl + \lfloor N\tilde{c}t \rfloor] = \frac{\hat{q}^{(N)}[Nl + \lfloor N\tilde{c}t \rfloor]}{N} \ . \tag{18}$$

It is well known that [8], in the limit of large N, the process $q^{(N)}(t)$ behaves like a deterministic process (in the interval under consideration). Let $A_c^{(N)}[k]$ denote the number of arrivals due to the controlled flows in slot k. Since $A_c^{(N)}[k] \sim$ Poisson$(x^{(N)}[N(l-d/2)]/\tilde{c})$ and also $x^{(N)}[N(l-d/2)] \stackrel{a.s.}{\to} \tilde{x}_{(l-d/2)}$ by the induction hypothesis, we have from a triangular array version of strong law of large numbers that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{\lfloor N\tilde{c}t \rfloor} A_c^{(N)}[k] \stackrel{a.s.}{=} \tilde{x}_{(l-\frac{d}{2})} t$$

Further since $q_l^{(N)}(0) = \overline{q}^{(N)}[Nl] \to \tilde{q}_l$ by the induction hypothesis, it follows from [8] that,

$$\lim_{N \to \infty} q_l^{(N)}(t) = q_l(t) \text{ u.o.c.,} \tag{19}$$

$$q_l(t) = (\tilde{q}_l + t(x + a - \tilde{c}))^+. \tag{20}$$

We thus have

$$\overline{q}^{(N)}[N(l+1)] = q_l^{(N)}(\tau) \to q_l(\tau) = \tilde{q}_{(l+1)} \ .$$

Further, note that, if $M_i^{'(N)}[Nk]$ denotes the number of packets of source i marked in the slot interval $[Nl, (N+1)l]$, we have

$$y_i^{(N)}[N(l+1)] = y_i^{(N)}[Nl] + \kappa \left[w - \frac{1}{\tau} M_i^{'(N)}[N(l+1-\frac{d}{2})] \right] \tag{21}$$

By the induction hypothesis, $y_i^{(N)}[Nl] \xrightarrow{d} \tilde{y}_l$ and by Theorem 3.4

$$M_i^{'(N)}[N(l+1-\frac{d}{2})] \xrightarrow{d} \text{Poisson}(\tau \tilde{y}_{(l-\frac{d}{2})} p_q(\tilde{q}_l, \tilde{x}_{(l-\frac{d}{2})}))$$

and so $y_i^{(N)}[N(l+1)] \xrightarrow{d} \tilde{y}_{(l+1)}$ by applying the continuous mapping theorem.

Since $y_i^{(N)}[N(l+1)] \xrightarrow{d} \tilde{y}_{(l+1)}$, we have

$$x^{(N)}[N(l+1)] = \frac{1}{N}\sum_{i=1}^{N} y_i^{(N)}[N(l+1)] \xrightarrow{a.s.} \mathbb{E}[\tilde{y}_l] \tag{22}$$

by the law of large numbers. The result is thus true for $(l+1)$. □

The equations given by (14)-(16) can be used to completely characterize $\tilde{x}_l = \mathbb{E}[\tilde{y}_l]$ and all the moments of $\tilde{y}_l$ for all l and also in steady state.

3.1.2 TCP-like controller

We now show similar models with a TCP-like controller. The equivalent marking function, $p_q(q, x)$, as a function of the normalized queue length at the boundaries of the rate update interval and the average flow rate of the controlled flows takes a similar form as in Section 3.1.1. However, the rate update equations are different in this case. The evolution of the system with a TCP-like controller can be written as follows for large N:

$$y_i^{(N)}[N(k+1)] - y_i^{(N)}[Nk] = \kappa\left[w - \frac{1}{\tau} y_i^{(N)}[Nk] M_i^{'(N)}[N(k+1-\frac{d}{2})]\right], \tag{23}$$

where $M_i^{'(N)}[Nk]$ denotes the number of packets of source i marked in the slot interval $[Nk, (N+1)k]$.

Theorem 3.3. *For a TCP-like controller and REM with $\gamma^{(N)} = \gamma/N$, as $N \to \infty$, $x^{(N)}[Nl] \xrightarrow{a.s.} \tilde{x}_l$, $\overline{q}^{(N)}[Nl] \xrightarrow{a.s.} \tilde{q}_l$, and $y_i^{(N)}[Nl] \xrightarrow{d} \tilde{y}_l$, where*

$$\tilde{y}_{(l+1)} - \tilde{y}_l = \kappa\left[w - \frac{1}{\tau}\tilde{y}_{(l+1-d)} M_{(l+1-\frac{d}{2})}\right] \tag{24}$$

$$M_l \sim \textit{Poisson}(\tau \tilde{y}_{(l-\frac{d}{2})} p_q(\tilde{q}_l, \tilde{x}_{(l-\frac{d}{2})})) \tag{25}$$

$$\tilde{x}_{(l+1)} - \tilde{x}_l = \kappa\left[w - \mathbb{E}[\tilde{y}_l \tilde{y}_{(l+1-d)}] p_q(\tilde{q}_{(l+1-\frac{d}{2})}, \tilde{x}_{(l+1-d)})\right] \tag{26}$$

$$\tilde{q}_{(l+1)} = (\tilde{q}_l + \tau(x_{(l-\frac{d}{2})} + a - \tilde{c}))^+ . \tag{27}$$

□

The model above is nearly identical to the one in [31] except for the fact that we distinguish between the queue length and number of packets in a window. This distinction is important when using this model to predict queue lengths accurately.

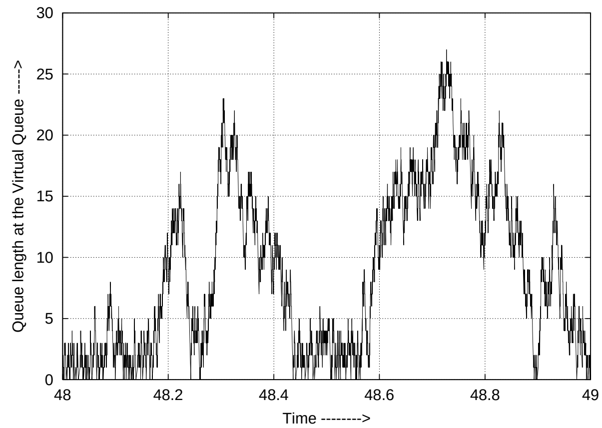

Figure 1: Virtual queue length with marking function $f(\hat{q}) = 1 - \exp(-0.0075\hat{q})$. $N = 25$ flows, $C = 2000$ packets/s, $RTT = 0.1$ s.

3.2 REM with $\gamma^{(N)} = \gamma$

We now consider a regime when $\gamma^{(N)}$ is not scaled with the number of flows in the system. We point out that, for a continuous time version of the problem (i.e. describing the rate updates by ODE's), and by neglecting the randomness due to the probabilistic marking, results similar to the ones obtained in this section are derived in [34]. The proof techniques are quite different in the case of the discrete time framework we are going to analyze.

First, we show some simulation results to indicate that the virtual-queue length process fluctuates a lot in this scaling-regime. The behavior of the virtual queue length in this regime for a particular choice of parameters is shown in Figure 1. Note that the virtual-queue length process hits zero many times in a slot (rate update interval). Thus, it is reasonable to expect that the behavior of the system cannot simply be described in terms of the queue-length at the boundaries of the rate-update intervals.

3.2.1 Proportionally fair controller

We argue that the appropriate model for the marking function in this parameter regime is a rate-based model. The following theorem provides the model for the marking function in this parameter regime.

Theorem 3.4. *Consider a time interval of length τ over which the rates of the controlled flows do not change. Further, let $y_i^{(N)}$, and $x^{(N)}$ be the rate of the i^{th} controlled flow into the link and the average rate of the N controlled flows into the links, respectively, at the beginning of the time interval under consideration.*

Also let $M_i^{(N)}$ be the random variable denoting the number of packets of the i^{th} controlled flow marked in this interval of length τ. If $y_i^{(N)} \stackrel{d}{\to} y_i$, $x^{(N)} \stackrel{a.s.}{\to} x$ as $N \to \infty$, then under Assumptions 1 and 2 and the parameter regime $\gamma^{(N)} = \gamma$, we have

$$\lim_{N\to\infty} M_i^{(N)} \stackrel{d}{=} \text{Poisson}(\tau y_i \overline{p}(x)) \tag{28}$$

where

$$\overline{p}(x) = 1 - \frac{\exp\left(\frac{x+a}{\tilde{c}}(e^{-\gamma}-1)\right)\left(1-\frac{x+a}{\tilde{c}}\right)^+(1-e^{-\gamma})}{\exp\left(\frac{x+a}{\tilde{c}}(e^{-\gamma}-1)\right)-e^{-\gamma}} \ . \tag{29}$$

Proof. In the proof we will use the notation $\rho = (x+a)/\tilde{c}$. Our starting point in the proof is similar to that of Theorem 3.4. Index the slots by $m, m+1, m+2 \dots m+N-1$. The virtual queue length process evolves as discrete-time queue with service rate one per slot and the number of arrivals in a slot distributed as Poisson($(x+a)/\tilde{c}$). Given $\hat{q}^{(N)}[k]$ and $y_i^{(N)}$, the random variable $M_i^{(N)}[k]$ is distributed as Poisson($f(\hat{q}^{(N)}[k] y_i^{(N)}/(N\tilde{c})$)). Here $f(\hat{q})$ is the marking probability which in in this parameter regime is given by

$$f(\hat{q}) = 1 - \exp(-\gamma\hat{q}) \ .$$

We are interested in finding the distribution of $M_i^{(N)} = \sum_{k=1}^{N\tilde{c}\tau} M_i^{(N)}[k]$.

Now, since the number of packets from source i in different slots are independent and Poisson with mean $y_i^{(N)}/(N\tilde{c})$, conditioned on the virtual-queue-length evolution process and $y_i^{(N)}$, the number of marked packets of source i is Poisson. Thus,

$$\begin{aligned} M_i^{(N)} \mid \{\hat{q}^{(N)}[k], k = m, m+1, m+2 \dots m+N-1\}, y_i^{(N)} &\sim \text{Poisson}\left(\frac{\sum_{k=m}^{m+N-1} f(\hat{q}^{(N)}[k]) y_i^{(N)}/(N\tilde{c}))}{N\tilde{c}}\right) \\ &= \text{Poisson}\left(y_i^{(N)} \tau S^{(N)}\left(\frac{x^{(N)}+a}{\tilde{c}}\right)\right) \end{aligned}$$

where $S^{(N)}((x^{(N)}+a)/\tilde{c}) = (1/N)\sum_k f(\hat{q}^{(N)}[k])$ and we have explicitly shown that the distribution of the random variable $S^{(N)}((x^{(N)}+a)/\tilde{c})$ depends on $(x^{(N)}+a)/\tilde{c}$ which is the mean number of arrivals in a slot of a discrete time $M/D/1$ queue. Note that, if $x^{(N)} = x$ (instead of $x^{(N)} \stackrel{a.s.}{\to} x$), it trivially follows from the ergodicity of a discrete time $M/D/1$ queue that $S^{(N)}(\rho)$ converges almost surely to $\mathbb{E}_{\pi(\rho)}[f(\hat{q}_\infty)]$, where $\pi(\rho)$ is the distribution of $\hat{q}_\infty$, the steady state queue length of an $M/D/1$ queue with arrival rate in each slot ρ. We will show that such this holds even when $x^{(N)} \stackrel{a.s.}{\to} x$.

Before, we go into the details of the proof, we make a quick note on the exact expression of $\mathbb{E}_{\pi(\rho)}[f(\hat{q}_\infty)]$. Note the following.

$$\tilde{\Pi}(z) = \mathbb{E}_{\pi(\rho)}[z^{\hat{q}_\infty}] = \frac{\exp(\rho(z-1))(1-\rho)(1-z)}{\exp(\rho(z-1))-z} \ . \tag{30}$$

The above expression can be used to compute $\mathbb{E}_{\pi(\rho)}[f(\hat{q}_\infty)]$ for any given marking function. In the case of REM, the equivalent marking profile $\overline{p}(x)$ is given by

$$\begin{aligned}\mathbb{E}_{\pi(\rho)}[f(\hat{q}_\infty)] &= \mathbb{E}_\pi[1-\exp(-\gamma\hat{q}_\infty)] \\ &= 1 - \frac{\exp\left(\rho(e^{-\gamma}-1)\right)(1-\rho)^+\left(1-e^{-\gamma}\right)}{\exp\left(\rho(e^{-\gamma}-1)\right)-e^{-\gamma}} \ .\end{aligned}$$

We will now show that $S^{(N)}((x^{(N)}+a)/\tilde{c})$ converges when $x^{(N)} \stackrel{a.s.}{\to} x$. To this end, we note that it is sufficient to prove the following lemma.

Lemma 3.1. *For any $\delta > 0$,*

$$\overline{p}(x-\delta) \le \liminf_{N\to\infty} S^{(N)}((x^{(N)}+a)/\tilde{c}) \le \limsup_{N\to\infty} S^{(N)}((x^{(N)}+a)/\tilde{c}) \le \overline{p}(x+\delta) \ \ a.s. \ ,$$

where

$$\overline{p}(x) \triangleq \begin{cases} \mathbb{E}_{\pi((x+a)/\tilde{c})}[f(\hat{q}_\infty)] & if \ \frac{x+a}{\tilde{c}} < 1 \\ 1 \qquad else. \end{cases} \tag{31}$$

Proof. Let $\delta > 0$ be an arbitrarily small number in this proof. Since, $x^{(N)} \stackrel{a.s.}{\to} x$, $\exists\ N_0(\delta)$ such that

$$x-\delta \le x^{(N)} \le x+\delta \ \ a.s. \ \forall N \ge N_0(\delta) \ .$$

For the rest of this proof, we will simply consider N such that $N \ge N_0(\delta)$ without explicitly mentioning it.

The proof relies on a few simple facts which we state without proof below. We use the notation $X \prec Y$ for two random variables to mean that Y stochastically dominates X, i.e., $\Pr(Y \ge z) \ge \Pr(X \ge z)$ for any real z.

Fact 1. *If A_1 and A_2 are two random variables with $A_1 \stackrel{d}{=}$ Poisson$((x_1+a)/\tilde{c})$, $A_2 \stackrel{d}{=}$ Poisson$((x_2+a)/\tilde{c})$, and $x_2 \le x_1$, then*

$$A_2 \prec A_1$$

Fact 2. *Let $q_i[k]$ ($i=1,2$) denote the queue-length process of a discrete time queueing system with one service per slot and the iid arrival process in each slot distributed as A_i ($i=1,2$). If $A_2 \prec A_1$, then $q_2[k] \prec q_1[k]\ \forall k \ge 0$, if $q_2[0] \prec q_1[k]$.*

We next introduce some notations. Let $q_{(\alpha)}[k]$ denote the queue-length of a discrete time $M/D/1$ queue with arrival in each slot Poisson(α). Combining Facts 1 and 2 we have for $N \ge N_0(\delta)$

$$q_{((x-\delta+a)/\tilde{c})}[k] \prec q_{((x^{(N)}+a)/\tilde{c})}[k] \prec q_{((x+\delta+a)/\tilde{c})}[k] \tag{32}$$

from which it readily follows that

$$f(q_{((x-\delta+a)/\tilde{c})}[k]) \prec f(q_{((x^{(N)}+a)/\tilde{c})}[k]) \prec f(q_{((x+\delta+a)/\tilde{c})}[k])$$

since $f(.)$ is an increasing function. This further implies

$$S^{(N)}((x-\delta+a)/\tilde{c}) \prec S^{(N)}((x^{(N)}+a)/\tilde{c}) \prec S^{(N)}((x+\delta+a)/\tilde{c}) \, . \tag{33}$$

We now consider two cases: $(x-\delta+a)/\tilde{c} < 1$, and $(x-\delta+a)/\tilde{c} > 1$. (we can ignore the case $(x-\delta+a)/\tilde{c} = 1$ by choosing δ small enough). First suppose $(x-\delta+a)/\tilde{c} < 1$. Using the ergodic-theorem, we have

$$\lim_{N\to\infty} S^{(N)}((x-\delta+a)/\tilde{c}) = \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(q_{((x-\delta+a)/\tilde{c})}[k]) \stackrel{a.s.}{=} \mathbb{E}_{\pi((x-\delta+a)/\tilde{c})}[f(\hat{q}_\infty)] \stackrel{\triangle}{=} \overline{p}(x-\delta), \tag{34}$$

in an *almost sure* sense, where $\pi((x-\delta+a)/\tilde{c})$ is the distribution of the steady state queue length of an $M/D/1$ queue with arrival rate in each slot $(x-\delta+a)/\tilde{c}$. On the other hand, if $(x-\delta+a)/\tilde{c} > 1$, it is an easy matter to show that

$$\lim_{N\to\infty} S^{(N)}((x-\delta+a)/\tilde{c}) = \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(q_{((x-\delta+a)/\tilde{c})}[k]) = 1 \stackrel{\triangle}{=} \overline{p}(x-\delta) \;\; a.s. \tag{35}$$

since $f(\infty) = 1$. Similarly

$$\lim_{N\to\infty} S^{(N)}((x+\delta+a)/\tilde{c}) = \overline{p}(x+\delta) \;\; a.s. \tag{36}$$

We thus have from (33), (35), and (36) that,

$$\overline{p}(x-\delta) \prec \lim\inf_{N\to\infty} S^{(N)}((x^{(N)}+a)/\tilde{c}) \prec \lim\sup_{N\to\infty} S^{(N)}((x^{(N)}+a)/\tilde{c}) \prec \overline{p}(x+\delta)$$

which immediately implies

$$\overline{p}(x-\delta) \le \lim\inf_{N\to\infty} S^{(N)}((x^{(N)}+a)/\tilde{c}) \le \lim\sup_{N\to\infty} S^{(N)}((x^{(N)}+a)/\tilde{c}) \le \overline{p}(x+\delta) \;\; a.s.$$

□

Since, Lemma 3.1 is true for arbitrarily small δ, and further $\overline{p}(x)$ is continuous for all $x \ge 0$, it follows that $S^{(N)}((x^{(N)}+a)/\tilde{c}) \stackrel{a.s.}{\to} \overline{p}(x)$. Thus it is also true that $S^{(N)}((x^{(N)}+a)/\tilde{c})$ converges in distribution. Now since

$$M_i^{(N)} | S^{(N)}((x^{(N)}+a)/\tilde{c}), y_i^{(N)} \sim \text{Poisson}(y_i^{(N)} \tau S^{(N)}((x^{(N)}+a)/\tilde{c})) \, ,$$

and $y_i^{(N)} \stackrel{d}{\to} y_i$, a straightforward manipulation yields that $M_i^{(N)}$ converges in distribution to Poisson$(y_i \tau \overline{p}(x))$.

□

We have the following result describing the complete behavior of the system.

Theorem 3.5. *For a proportionally-fair controller and REM with $\gamma^{(N)} = \gamma$, as $N \to \infty$, $x^{(N)}[Nl] \stackrel{a.s.}{\to} \tilde{x}_l$, and $y_i^{(N)}[Nl] \stackrel{d}{\to} \tilde{y}_l$, where*

$$\tilde{x}_{(l+1)} - \tilde{x}_l = \kappa \left[w - \tilde{x}_{(l+1-d)} \overline{p}(\tilde{x}_{(l+1-d)}) \right] \tag{37}$$

$$\tilde{y}_{(l+1)} - \tilde{y}_l = \kappa \left[w - \frac{1}{\tau} M_l \right] \tag{38}$$

$$M_l \sim \textit{Poisson}(\tau \tilde{y}_{(l+1-d)} \overline{p}(\tilde{x}_{(l+1-d)})) \tag{39}$$

Proof. The proof is similar to the proof of Theorem 3.3. □

The above theorem can be used to completely characterize $\tilde{x}_l = \mathbb{E}[\tilde{y}_l]$ and all the moments of $\tilde{y}_l$ for all l and also in the steady state.

3.2.2 TCP-like controller

The equivalent marking function, $\overline{p}(x)$, as a function of the the average flow rate of the controlled flows takes a similar form.

Theorem 3.6. *For a TCP-like controller and REM with $\gamma^{(N)} = \gamma$, as $N \to \infty$, $x^{(N)}[Nl] \stackrel{a.s.}{\to} \tilde{x}_l$ and, $y_i^{(N)}[Nl] \stackrel{d}{\to} \tilde{y}_l$, where*

$$\tilde{y}_{(l+1)} - \tilde{y}_l = \kappa \left[w - \frac{1}{\tau} \tilde{y}_{(l+1-d)} M_{(l-\frac{d}{2})} \right] \tag{40}$$

$$\tilde{x}_{(l+1)} - \tilde{x}_l = \kappa \left[w - \mathbb{E}[\tilde{y}_l \tilde{y}_{(l+1-d)}] \overline{p}(\tilde{x}_{(l+1-d)}) \right] \tag{41}$$

$$M_l \sim \textit{Poisson}(\tau \tilde{y}_{(l-\frac{d}{2})} \overline{p}(\tilde{x}_{(l-\frac{d}{2})})) \tag{42}$$

□

4 Equilibrium and Stability of Mean Rate with Proportionally Fair Controller

In this section we study the mean limiting behavior $\tilde{x}_l$ in the two parameter regimes. We will show that the two models have radically different stability criteria.

In the regime $\gamma^{(N)} = \gamma/N$, recall that the difference equation describing the evolution of the mean rate is as follows:

$$\tilde{x}_{(l+1)} - \tilde{x}_l = \kappa \left[w - \tilde{x}_{(l+1-d)} p_q(\tilde{q}_{(l+1-\frac{d}{2})}, \tilde{x}_{(l+1-d)}) \right]$$
$$\tilde{q}_{(l+1)} = (\tilde{q}_l + \tau(\tilde{x}_{(l-\frac{d}{2})} + a - \tilde{c}))^+ \ .$$

It immediately follows that the equilibrium point (x^*, q^*) is given by

$$x^* = \tilde{c} - a = \theta c - a, \ \ w = x^* p(q^*)$$

since $p_q(\tilde{q}_l, \tilde{c} - a) = p(\tilde{q}_l)$ (in the case of REM $p(\tilde{q}_l) = 1 - \exp(-\gamma \tilde{q}_l)$). However, the equilibrium can be reached only if the delay-difference equation is stable. This depends on the choice of γ. For a given value of γ, one can proceed with the difference equation and verify the stability of a linearized model numerically. However, to have a better insight on how we should choose γ to ensure stability, we derive conditions for a continuous time deterministic model to be stable. If the step size of the discrete-time model is small, then we can expect the conditions derived from this model to be reasonably accurate. To this end, consider the system given by

$$\frac{dx(t)}{dt} = k[w - x(t-r) p_q(\tilde{q}(t-r), x(t-r))]$$
$$\frac{d\tilde{q}(t)}{dt} = (x(t) + a - \theta c) I_{(\tilde{q}(t)>0)} + \ (x(t) + a - \theta c)^+ I_{(\tilde{q}(t)=0)}$$

In the above, $x(t)$ is a continuous time version of $\tilde{x}_l$ and k can be viewed as $k = \kappa/\tau$. We now find conditions for the local stability of the system. Linearizing the above system and denoting by $u(t)$ the shifted rate $(u(t) - x^*)$, and by $v(t)$ the shifted normalized virtual queue length $(\tilde{q}(t) - q^*)$ we get,

$$\frac{du(t)}{dt} = -kp_1(q^*, x^*) u(t-r) - \ kx^* p_2(q^*) v(t-r) \tag{43}$$
$$\frac{dv(t)}{dt} = u(t) \ . \tag{44}$$

In the above,

$$p_1(q^*, x^*) = 1 - \exp(-\gamma q)(1 - \tfrac{\gamma x^*}{2\tilde{c}})$$

and $p_2(q^*) = \gamma \exp(-\gamma q^*)$. We now provide condition for the system to be stable. The outline of the derivation using the Nyquist criterion is given in the Section A. A simplified stability criterion can be obtained, if we also suppose that $x^* r \geq p_1(q^*, x^*)/(\gamma \exp(-\gamma q^*))$. This means that the equilibrium window size is not too small. Then, the system given by (43)-(44) is stable if the parameters satisfy

$$(krp_1(q^*, x^*))^2 + \frac{(kx^* p_2(q^*) r^2)^2}{4\pi^2} \leq 4\pi^2 \ ,$$

and the system is unstable if

$$(krp_1(q^*))^2 + \frac{4(kx^* p_2(q^*)r^2)^2}{81\pi^2} \geq \frac{81\pi^2}{4} \ .$$

Note that, in so far as $p_1(q^*, x^*)$ is very small, the stability condition is roughly equivalent to

$$kx^* r^2 \gamma < 4\pi^2 \ . \tag{45}$$

Thus, stability requires that k and γ both to be inversely scaled with r. Further, note that the stability condition is equivalently

$$x^* < \frac{4\pi^2}{kr^2\gamma} \tag{46}$$

which implies that the system is stable for small target rate allocation per user.

We now consider the parameter regime $\gamma^{(N)} = \gamma$ where the equivalent marking function is a rate based one. We have the following difference equation for $\tilde{x}_l$:

$$\tilde{x}_{(l+1)} - \tilde{x}_l = \kappa \left[w - \tilde{x}_{(l+1-\frac{d}{2})} \overline{p}(\tilde{x}_{(l+1-\frac{d}{2})}) \right] \ . \tag{47}$$

It immediately follows that the equilibrium point x^* is given by the solution of

$$w = x^* \overline{p}(x^*) \ .$$

One can obtain the stability condition of the above system numerically. However, a simplified condition for the local-stability can be obtained from the corresponding continuous-time version of $\tilde{x}_l$ which evolves as follows:

$$\frac{dx(t)}{dt} = k[w - x(t-r)\overline{p}(x(t-r))] \ .$$

A sufficient condition for the local stability [19] of the above system is given by

$$kr(\overline{p}(x^*) + x^* \overline{p}^{'}(x^*)) \leq \frac{\pi}{2} \ .$$

Further, it can be shown that, for $x^* < \tilde{c} - a$, $x^* \overline{p}'(x^*)/\overline{p}(x^*) \leq K_1$ for a suitable K_1 which depends on γ. Thus, a sufficient condition for stability is $kr\overline{p}(x^*) \leq \pi/(2(1+K_1)$. Since $x^*\overline{p}(x^*) = w$, the stability condition is equivalently

$$x^* \geq \frac{2krw(1+K_1)}{\pi}. \tag{48}$$

Thus the system is stable for large target rate allocation per source, which is in sharp contrast to the stability criterion given by (46) in the parameter regime $\gamma^{(N)} = \gamma/N$. The analysis in this section shows that, unless the appropriate limiting model is used, one can obtain entirely different conditions for system stability.

5 Simplifications with TCP-Like Controller for Calculating Equilibrium

In the previous section, we presented models to characterize the the behavior of the average and the individual flow rate. In the case of proportionally fair controller, the equilibrium point is easy to compute from the limiting behavior of the mean flow rate. However, in the case of TCP-like controller the equilibrium point of the mean rate is not easy to compute. Note that, in the regime $\gamma^{(N)} = \gamma/N$ the mean rate can be described as follows:

$$\tilde{x}_{(l+1)} - \tilde{x}_l = \kappa \left[w - \mathbb{E}[\tilde{y}_l \tilde{y}_{(l+1-d)}] p_q(\tilde{q}_{(l+1-\frac{d}{2})}, \tilde{x}_{(l+1-d)}) \right] \ .$$

Clearly, calculating the equilibrium is quite complicated and can only be done along with system of stochastic equations describing the update of $\tilde{y}_l$. Thus, as in [28, 7], we make an additional assumption which leads to simple difference equations describing the behavior of the mean rate. We assume $\tilde{y}_l$ and $\tilde{y}_{(l+1-d)}$ to be uncorrelated so that we can replace the term $\mathbb{E}[\tilde{y}_l \tilde{y}_{(l+1-d)}]$ by $\tilde{x}_l \tilde{x}_{(l+1-d)}$ in (26) and (41). The resulting deterministic equations (which we describe shortly) are also known as deterministic fluid models. Such models have been widely used to study the equilibrium and stability properties of congestion controlled sources and have been shown to predict the mean rate of the system very accurately [23, 18, 19, 32, 26].

In the parameter regime $\gamma^{(N)} = \gamma/N$, we thus have the following difference equation describing the mean behavior x_l if we substitute $\tilde{x}_l \tilde{x}_{(l+1-d)}$ for $\mathbb{E}[\tilde{y}_l \tilde{y}_{(l+1-d)}]$ in (26):

$$\tilde{x}_{(l+1)} - \tilde{x}_l = \kappa \left[w - \tilde{x}_l \tilde{x}_{(l+1-d)} p_q(\tilde{q}_{(l+1-\frac{d}{2})}, \tilde{x}_{(l+1-d)}) \right]$$
$$\tilde{q}_{(l+1)} = (\tilde{q}_l + \tau(\tilde{x}_{(l-\frac{d}{2})} + a - \tilde{c}))^+ \ .$$

It immediately follows that the equilibrium point (x^*, q^*) is given by

$$x^* = \tilde{c} - a = \theta c - a, \ \ w = {x^*}^2 p(q^*) \ .$$

We now consider the parameter regime $\gamma^{(N)} = \gamma$. Again, if we substitute $\tilde{x}_l \tilde{x}_{(l+1-d)}$ for $\mathbb{E}[\tilde{y}_l \tilde{y}_{(l+1-d)}]$ in (41) we get the following:

$$\tilde{x}_{(l+1)} - \tilde{x}_l = \kappa \left[w - \tilde{x}_l \tilde{x}_{(l+1-d)} \overline{p}(\tilde{x}_{(l+1-d)}) \right] \ .$$

It immediately follows that the equilibrium point x^* is given by the solution of

$$w = {x^*}^2 \overline{p}(x^*) \ .$$

One can also obtain stability conditions with the equations describing the approximate behavior of the average rate.

6 Results and Discussion

6.1 Simulation setup

In this section, we show packet-based simulation results to validate some of our observations and results. The purpose of the simulations is the following. First, we demonstrate that the throughput as predicted by the suitable models for marking function with the different scaling of parameter is close to that observed from the simulations. Second, we demonstrate that one can indeed achieve a very small queueing delay at the link, provided the parameters are chosen appropriately.

We simulate a single bottleneck link accessed by multiple TCP sources, all of which are in the congestion avoidance phase. Apart from the TCP sources we also consider unresponsive flows. We use an ON-OFF model for the uncontrolled flows [17]. The uncontrolled flows toggle between ON and OFF state which are exponentially distributed with mean 0.2 s. In the ON state, an uncontrolled flow sends data at a rate ρ packets/s. In all our simulations with various AQM schemes, we change N, the number of TCP sources, which is also the number of uncontrolled flows in the system. The link capacity in all our simulations is Nc, where $c = 80$ packets/s. The flow rate ρ of the uncontrolled flows in the ON state is adjusted so that uncontrolled flows deliver a load of 25% into the link. Every simulation result is averaged over 10 runs.

As we have discussed in the beginning of this section, we propose to demonstrate that, to obtain a small queueing delay at the router, the parameters of the AQM scheme has to be chosen suitably. To this end, we report simulation results with four sets of parameters as follows:

1. $\theta = 0.85$, $\gamma^{(N)} = 0.0075/N$
2. $\theta = 0.85$, $\gamma^{(N)} = 0.0075$
3. $\theta = 1$, $\gamma^{(N)} = 0.05/N$
4. $\theta = 1$, $\gamma^{(N)} = 0.05$

6.2 Results

We first show results for the case $\theta = 0.85$, i.e., when the capacity of the virtual queue is $0.85Nc$, N being the number of TCP flows in the system. We compare the average throughput obtained from the simulation with the predicted equilibrium of the suitable limiting model for two different parameter scalings of REM: $\gamma^{(N)} = 0.0075/N$ and $\gamma^{(N)} = 0.0075$. The plots are shown in Figure 2. The equilibrium point of the suitable limiting models predict the average throughput into the link reasonably accurately. Further, if the capacity of the virtual queue is 0.85 fraction of the link capacity, it is possible to attain a mean queue-length (at the real-queue) that does not grow with N, and thus, providing a queueing delay of $O(1/N)$. Such a behavior can be observed in the both the regimes of $\gamma^{(N)}$ considered in the plots and this paper. Further, the plots for coefficient of variation justify Assumption 1.

Recall that our theoretical model implicitly assumes that $\theta < 1$ since we neglect variations in the RTT due to variations in the queueing delay. Thus, it is useful to consider the implications of choosing $\theta = 1$ through simulations. Observe that $\theta = 1$ is equivalent to marking packets based on the occupancy of the real-queue. In Figure 3, we show the plots of average throughput at the link and the mean queue-length for $\theta = 1$ with two different parameters scalings of $\gamma^{(N)}$: $\gamma^{(N)} = 0.05/N$ and $\gamma^{(N)} = 0.05$. Note that, in this case, an $O(1)$ queue length (and thus a queueing delay of $O(1/N)$) at the real-queue is obtained only in the parameter regime $\gamma^{(N)} = 0.05$.

Based on the two sets of plots, we summarize our observations as follows:

- If the capacity of the virtual queue is less than that of the link capacity, it is possible to attain a negligible queueing delay in the either parameters regimes of $\gamma^{(N)}$. The limiting models as obtained in the previous sections quite accurately predict the equilibrium values.

- If the capacity of the virtual queue is identical to the link capacity, simulations suggests that negligible queueing delay can be obtained only in the parameter regime $\gamma^{(N)} = \gamma$. In this case, the appropriate limiting model is a rate-based marking model even though marking may be implemented based on the contents of the queue.

7 Concluding Remarks

In this paper, we have provided appropriate models for virtual-queue based marking, based on the scaling of parameters in a virtual-queue based AQM scheme. Using virtual queue based REM as an illustration, we have shown that the limiting marking model when the number of flows is large can be wither a rate-based or virtual-queue length based model depending on the scaling of a certain parameter with the number of flows. While we have only shown convergence to a deterministic model in this paper, our stochastic model can also be used to calculate the variance around the deterministic limit. However, these calculations presented in [11] are somewhat heuristic in nature and a possible direction for future work is to rigorously justify them.

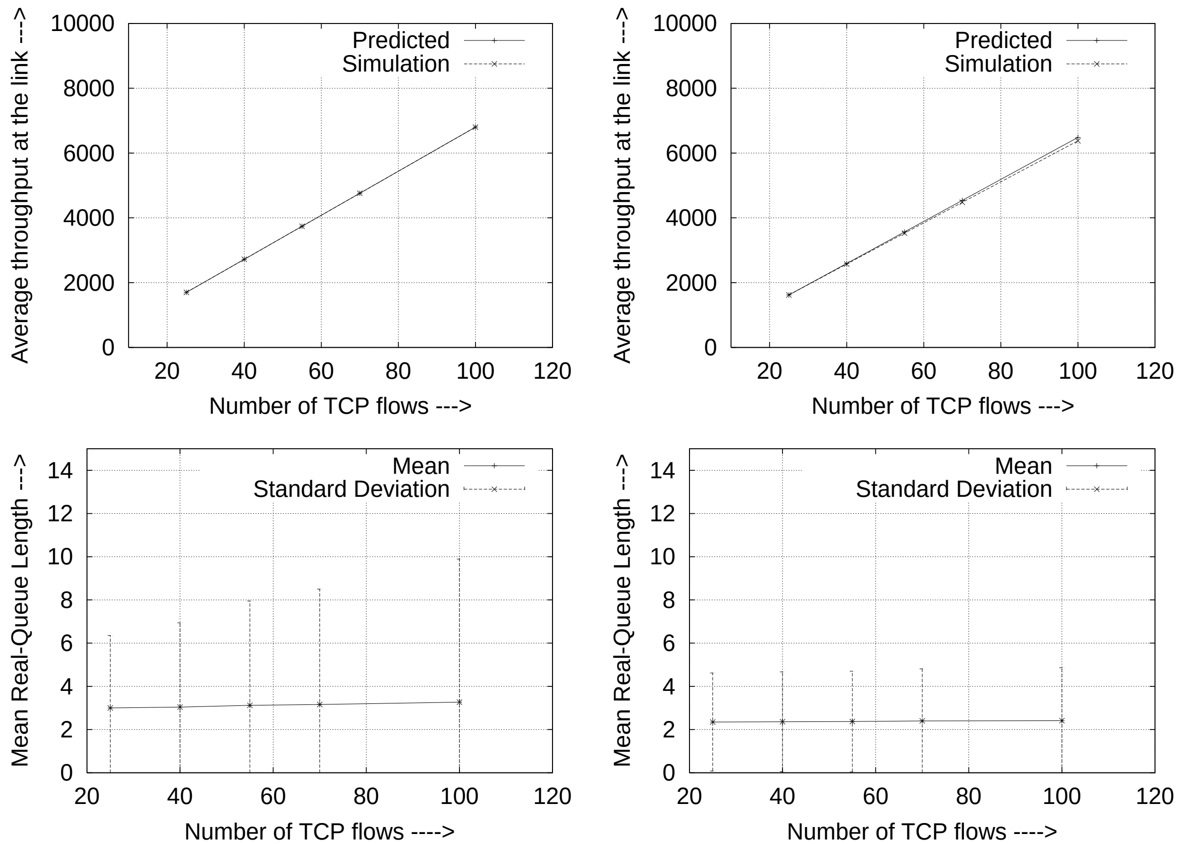

Figure 2: Comparison of average utilization, coefficient of variation and mean queue length with virtual queue based REM. On the left-hand panel we show plots when $\gamma^{(N)}$ is scaled as $\gamma^{(N)} = 0.0075/N$, and the right-hand panel shows plots with $\gamma^{(N)} = 0.0075$.

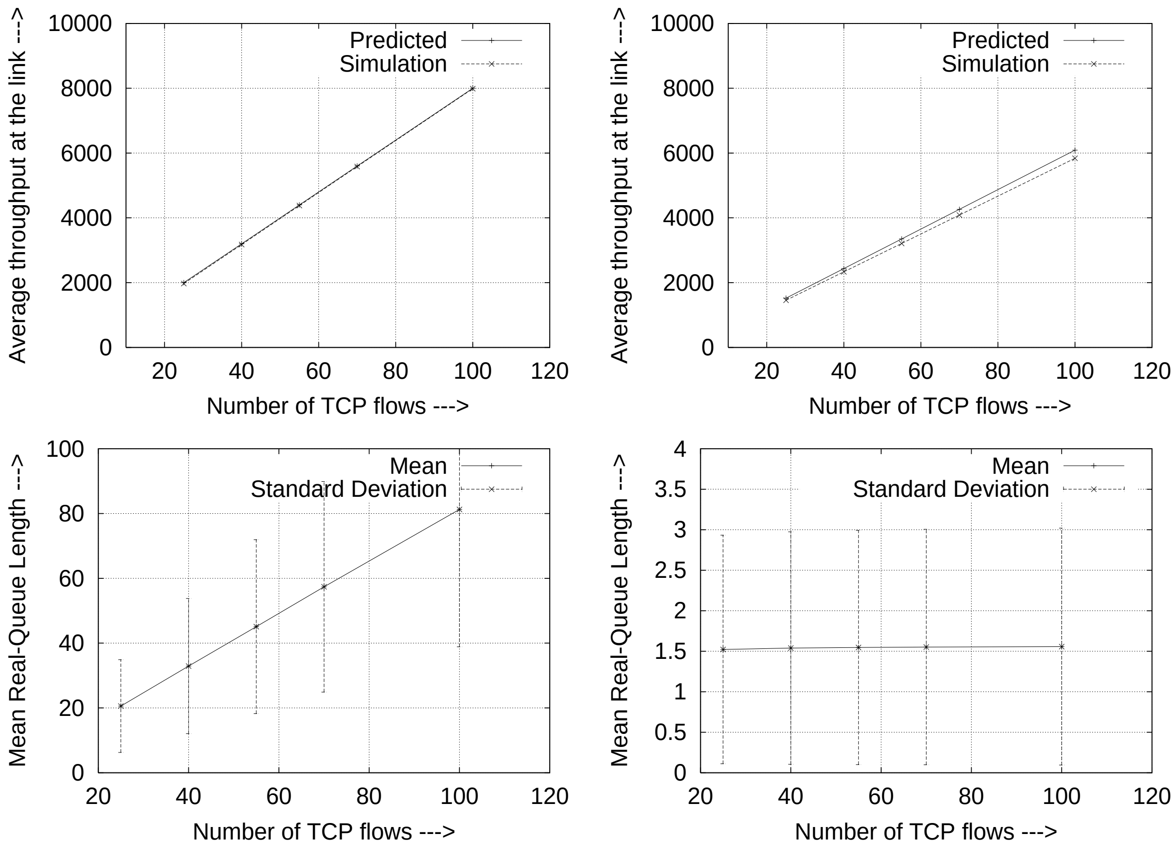

Figure 3: Comparison of average throughput, and mean queue length with virtual queue based REM with $\theta = 1$. On the left-hand panel we show plots when $\gamma^{(N)}$ is scaled as $\gamma^{(N)} = 0.05/N$, and the right-hand panel shows plots with $\gamma^{(N)} = 0.05$.

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A Stability Condition of Virtual Queue-Based REM with $\gamma^{(N)} = \gamma/N$

We now outline the derivation of the stability condition for the system given by (43)-(44). Note that the system under consideration can be expressed as

$$\begin{aligned} \frac{du}{dt} &= -au(t-r) - bv(t-r) \\ \frac{dv}{dt} &= u(t) \,. \end{aligned}$$

The above system is stable if the roots of

$$\begin{aligned} G(s) &= \det \begin{bmatrix} s + a\exp(-sr) & b\exp(-sr) \\ -1 & s \end{bmatrix} \\ &= s^2 \left(1 + \frac{a\exp(-sr)}{s} + \frac{b\exp(-sr)}{s^2} \right) \end{aligned}$$

lie in the left half of the complex plane. From the Nyquist stability criterion [29], it is sufficient to show that the Nyquist plot of $L(j\omega)$, where

$$L(s) = \frac{a\exp(-sr)}{s} + \frac{b\exp(-sr)}{s^2}$$

does not enclose the point $(-1 + j0)$. Expanding $L(j\omega)$, it can be verified that a sufficient condition for this is

$$h(\omega) = \frac{a\sin(\omega r)}{\omega} + \frac{b\cos(\omega r)}{\omega^2} \leq 1$$

whenever

$$\tan(\omega r) = \frac{a\omega}{b}, \;\; \omega \neq 0 \; . \tag{49}$$

The above is a sufficient condition for stability of the linear system under consideration.

The condition can be simplified if we further assume $x^* r \geq p_1(q^*, x^*)/(\gamma \exp(-\gamma q^*))$, or equivalently $a/(br) \leq 1$.

Now suppose $a/(br) < 1$. It is easy to see ω satisfying (49) is such that $\omega r \in [\pi, 3\pi/2] \cup [2\pi, 5\pi/2] \cup [3\pi, 7\pi/2] \ldots$. First note that if $\omega r \in [\pi, 3\pi/2] \cup [3\pi, 7\pi/2] \cup [5\pi, 11\pi/2] \ldots$, the values of $\cos(wr)$ and $\sin(\omega r)$ are negative, and so the condition $h(\omega) < 1$ for ω satisfying (49) is trivially satisfied. It is thus enough to consider ωr satisfying (49) in the range $\{[2\pi, 5\pi/2] \cup [4\pi, 9\pi/2] \ldots\}$ when the values of $\cos(wr)$ and $\sin(\omega r)$ are positive. Using routine trigonometric manipulations, it can be shown that, under (49), $h(\omega)$ satisfies

$$h(\omega) = \frac{\sqrt{a^2\omega^2 + b^2}}{\omega^2} \; .$$

Since $h(\omega)$ is decreasing in ω, by considering the solution of (49) such that $\omega r \in [2\pi, 5\pi/2]$, it follows that a sufficient condition for local stability is

$$a^2 r^2 + \frac{b^2 r^4}{4\pi^2} \leq 4\pi^2 \; .$$

We now find a sufficient condition for the instability of the linearized system when $a/(br) \leq 1$. First note that, for ωr satisfying (49) in the range $[2\pi, 5\pi/2] \cup [4\pi, 9\pi/2]$,

$$h(\omega) \geq \sqrt{\frac{4a^2 r^2}{81\pi^2} + \frac{16 b^2 r^4}{(81\pi^2)^2}} \; . \tag{50}$$

Thus, $h(\omega) > 1$ if

$$a^2r^2 + \frac{4b^2r^4}{81\pi^2} \geq \frac{81\pi^2}{4} \; . \tag{51}$$

Further, we also have ωr satisfying (49) in the range $[\pi, 3\pi/2] \cup [3\pi, 7\pi/2] \cup \ldots$ trivially satisfies $h(\omega) < 1$. Suppose, $\omega_2 r \in [2\pi, 5\pi/2]$, $\omega_3 r \in [3\pi, 7\pi/2]$, $\omega_4 r \in [4\pi, 9\pi/2]$, and $\omega_5 r \in [5\pi, 11\pi/2]$ satisfy (49). By our preceding argument, $h(\omega_2) > 1$, $h(\omega_3) < 1$, $h(\omega_4) > 1$ and $h(\omega_5) < 1$. It is not hard to see that the Nyquist plot of $L(j\omega)$ encircles $(-1 + j0)$, and hence, (51) provides a sufficient condition for instability of the linearized system.