Distributed Fair Resource Allocation in Cellular Networks in the Presence of Heterogeneous Delays

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Abstract

We consider the problem of allocating resources at a base station to many competing flows, where each flow is intended for a different receiver. The channel conditions may be time-varying and different for different receivers. It has been shown in [6] that in a delay-free network, a combination of queue-length-based scheduling at the base station and congestion control at the end users can guarantee queue-length stability and fair resource allocation. In this paper, we extend this result to wireless networks where the congestion information from the base station is received with a feedback delay at the transmitters. The delays can be heterogenous (i.e., different users may have different round-trip delays) and time-varying, but are assumed to be upper-bounded, with possibly very large upper bounds. We will show that the joint congestion control-scheduling algorithm continues to be stable and continues to provide a fair allocation of the network resources.

I. Introduction

We study the problem of fair allocation of resources in the downlink of a cellular wireless network consisting of a single base station and many receivers (see Figure 1). The data destined for each receiver is maintained in a separate buffer. The arrivals to the buffers are determined via a congestion control mechanism, which will be described in detail later. We assume that the time is slotted. The channels between the base station and the receivers are assumed to have random time-varying gains which are independent from one time-slot to the next. The independence assumption can be relaxed easily, but we use it here for ease of exposition. The goal is to allocate the network capacity fairly among the users, in accordance with the needs of the users,

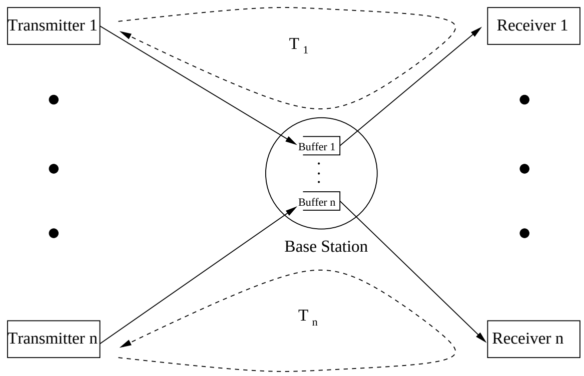

Fig. 1. Network with feedback delays. The channel from the base station to the receivers is time-varying.

while exploiting the time-variations in the channel conditions. We associate a utility function with each user that is a concave, increasing function of the mean service that it receives from the network. In an earlier paper [6], it was shown that a combination of Internet-style congestion control at the end-users and queue-length based scheduling at the base station achieves the goal of fair and stabilizing resource allocation. This result is somewhat surprising since the resource constraints in the case of a wireless network are very different from the linear constraints in the case of the Internet [17]. The relative merits of congestion control-based resource allocation scheme as compared to other resource allocation schemes for cellular networks are discussed in [6]. Several other works in the same context are [18], [10], [13]. However, none of these works explicitly include the effect of feedback delay in their analysis. In this work, we aim to consider the effect of this essential parameter on the fairness and stability properties of the algorithm presented in [6].

In [6], it is assumed that there are no delays in the transmission of packets from an end-user to the base station and in the transmission of congestion information from the base station back to the end users. But if we consider the case such that end users are connected to the base station through a multi-hop network or Internet, then delays exist in both directions: there is a propagation delay τ_i^f from the source i to the base station — we call it the forward delay of user i, and a propagation delay τ_i^b from the base station to the user i — we call it the backward delay. It is well-known that the presence of delays may affect the performance of the network. For example, Internet congestion controllers which are globally stable for the delay-free network may become unstable if the feedback delays are large [17]. In our problem, when delays exist, the information the end users obtain will be "outdated" information. So the congestion information the users obtain at time t does not reflect the queue status at the base station at time t. So it is interesting to study a wireless network with delays and ask whether the conclusions of [6] still hold for wireless networks with heterogeneous delays. We answer this question by showing that for a network with uniformly-bounded delays, which are potentially heterogeneous and time-varying, the algorithm of [6] is stable and can be used to approximate weighted-m fair allocation arbitrarily closely. We emphasize that the results hold for networks with arbitrarily large, but bounded time-varying delays. So even if the end users can only get very old feedback information from the base station, the network is still stable and can eventually reach the fair resource allocation. On the other hand, from the proof, we can also see that when the delays are large, it may take more time for the network to achieve the fair resource allocation. This observation is also supported by simulations, not shown here due to page limitation, which are presented in [21].

The rest of paper is organized as follows: in Section II, we introduce the system model including the congestion controller used by the end users and the scheduler implemented at the base station. In Section III, we show that the resulting resource allocation approximates weighted-m fairness arbitrarily closely. Finally, we conclude in Section IV.

II. System Model

We consider a cellular network shared by n flows in the downlink and assume that the base station maintains n separate queues, one corresponding to each flow. We study the fair resource allocation problem in this paper. Specifically we consider weighted m-fairness. It means that each source i has a utility function given by $U_i(\overline{z_i}) = \alpha_i \frac{\overline{z_i}^{1-m}}{1-m}$, where $\overline{z_i}$ is the average rate at which user i transmits and α_i is a positive weighting factor [12]. Here, $m = 1$, $m = 2$ and $m \to \infty$ correspond to respectively proportional fair, minimum potential delay fair and max-min fair resource allocations. We assume that the time is slotted and denote the length of the queue i at the beginning of the time slot t by $x_i[t]$, the number of arrivals to queue i in time slot t by $a_i[t]$, and the amount of service offered to queue i in slot t by $\mu_i[t]$. We assume that each of these parameters can only take non-negative integer values. The evolution of the size of the ith queue is given by:

$$x_i[t+1] = x_i[t] + a_i[t] - \mu_i[t] + u_i[t],$$

where $u_i[t]$ is a non-negative quantity which denotes the wasted service given to queue i at time slot t and it guarantees that $x_i[t] \geq 0$. We also assume that the channel between the base station and the receivers can be in one of J states in a given slot. We use $s[t]$ to denote the state in time slot t. The channel state is assumed to be fixed within a time slot, but may vary from one slot to another, thus capturing the time-varying characteristics of a fading environment. Corresponding to each channel state, say j, is an achievable rate region, C_j, that is defined to be convex hull of the feasible rate vectors, $\eta[t] := (\eta_1[t], \ldots, \eta_n[t])$, that can be offered to the queues. We assume that each C_j is a bounded region and let $\hat{\eta} < \infty$ denote the upper bound on the achievable rates for all channel states. The channel state process is assumed to be independent and identically distributed in each time slot, but we do not require that the statistics be known at the base station. Furthermore, we define the mean achievable rate region as

$$\bar{C} := \left\{ \eta : \eta = \sum_{j=1}^{J} \pi_j^{ch} \eta^{(j)}, \eta^{(j)} \in C_j \right\},$$

where π_j^{ch} stands for the stationary distribution of the channel state process being in state j. The scheduler will use following algorithm:

SCHEDULER: Given the current queue length $\mathbf{x}[t] := (x_1[t], \ldots, x_n[t])$ and current channel state $s[t]$, the scheduler chooses a service rate vector $\mu[t] := (\mu_1[t], \ldots, \mu_n[t]) \in C_{s[t]}$ that satisfies

$$\mu[t] \in \arg\max_{\mu \in C_{s[t]}} \sum_{i=1}^{n} x_i[t] \eta_i.$$

This scheduling rule was introduced in the context of fixed arrival rates (i.e., where the arrival rates are not adjusted by a congestion controller) in [19], where it was also shown that it is a stabilizing rule, i.e., the mean queue-lengths are upper-bounded. This result was extended in many different directions in [2], [16], [15], [7], [3], [9], [5], [14].

In our model, the packet arrival rate into the queue is assumed to be controlled according to the well-known dual controller that has been studied extensively in the context of Internet congestion control [8], [11], [20], [17]. In the context of Internet congestion control, a dual controller chooses the transmission rate z_i such that

$$\frac{\alpha_i K}{x_i} = z_i$$

for any constant $K > 0$. Next, we describe the operation of our congestion controller followed by some assumptions.

CONGESTION CONTROLLER In our wireless network model, accounting for the forward delay τ_i^f and the backward delay τ_i^b, the mean of the number of arrivals into queue i, given t, is:

$$E[a_i[t]|x_i[t-T_i]] = \min\left\{\frac{\alpha_i K}{(x_i[t-T_i])^m}, M\right\}, \tag{1}$$

where $m > 0$, $T_i = \tau_i^f + \tau_i^b$ and $M > 2\hat{\eta}$ is a positive constant which ensures that the arrival rate into the queue is upper bounded when the queue length is small. We will later show that K plays a critical role in the context of a cellular network: it is important to choose K to be large to ensure weighted m-fair resource allocation. The parameter K does not play such a role in the context of wireline networks. We assume $a_i[t]$ is independent across time slots and that the variance of $a_i[t]$ given $x_i[t-T_i]$ is bounded:

$$E\left[a_i^2[t]|x_i[t-T_i]\right] \le V < \infty \quad \text{for all } x_i[t-T_i]. \tag{2}$$

Furthermore, we assume there exist positive numbers θ, $A > T\hat{\eta}$ and $h > 2$ such that for any $N > A$,

$$p(\sum_{j=1}^{T} a_i[t-j] = N) < \frac{\theta}{N^h} \text{ for all } i. \tag{3}$$

We will later show that K plays a critical role in the context of a cellular network: it is important to choose K to be large enough to ensure weighted proportionally-fair resource allocation. The parameter K does not play such a role in the context of wireline networks.

In summary, the combined Scheduler-Congestion Controller Algorithm can be defined as follows:

$$x_i[t+1] = x_i[t] + a_i[t] - \mu_i[t] + u_i[t] \tag{4}$$

$$\mu[t] \in \arg\max_{\mu \in C_{s[t]}} \sum_{i=1}^{n} x_i[t]\eta_i, \tag{5}$$

where $a_i[t]$ is a random variable satisfying the conditions in (1), (2) and (3). Note that the congestion control part of this algorithm is slightly different from the algorithm in [6]. We impose an upper-bound on the source rates in a more natural manner than in [6]. Our results continue to hold for the algorithm in [6] too.

We now present the following theorem, which will be useful later. This theorem is similar to Proposition 1 of [6].

Theorem 1: There exists a unique pair of vectors $(\mathbf{x}^*, \mu^*)$ which satisfy following conditions

- $\mu^* \in \arg\max_{\eta \in \bar{C}} \sum_{i=1}^n x_i^* \eta_i; \quad x_i^* = \left(\frac{\alpha_i K}{\mu_i^*}\right)^{\frac{1}{m}}$ for all i, and
- μ^* is the optimal solution to $\max_{\mu \in \bar{C}} \sum_{i=1}^n K \alpha_i \frac{\mu_i^{1-m}}{1-m}$.

◇

From the above theorem, we can see that μ^* is weighted-m fair. For the stochastic model, we will show that $\mathbf{x}[t]$ converges to $\mathbf{x}^*$, defined in Theorem 1, in a probabilistic sense. This then implies that the network reaches a fair operating point.

In the rest of the paper, we will show that fair resource allocation can be achieved when the linear gain, K, used in the congestion controller goes to infinity. The proof is somewhat involved, but the idea behind the proof is relatively simple. From [6], it has been known that the result holds for a delay-free network. In that paper the authors define a quadratic Lyapunov function, which we denote as $W_0(t)$, and prove that for large K,

$$E[W_0(t+1) - W_0(t) | W_0(t)] \le G(K) < 0,$$

where $G(\cdot)$ is a properly defined function. Here, we will use a similar Lyapunov function that incorporates the feedback delay. Let us call this Lyapunov function $W(t)$. We show that

$$E[W(t+1) - W(t) | W(t)] \le G(K) + H(K),$$

where $H(K)$ is an additional term due to the delays. We further prove that $H(K)/G(K) \to 0$ as $K \to \infty$, which implies that for large K,

$$E[W(t+1) - W(t) | W(t)] \le G(K)/2 < 0.$$

Thus, following the arguments of [6], we can show that the fair resource allocation can be achieved for large K.

III. Positive recurrence and weighted-proportional fairness

The wireless network is a discrete-time stochastic system, so we have to show the stochastic stability of this system, i.e., we have to show that a Markov chain describing the evolution of this system is positive recurrent. We define $\mathbf{y}[t] = (\mathbf{x}[t], \ldots, \mathbf{x}[t-T])$, where $T = \max_i T_i$. It is easy to see that the process $\{\mathbf{y}[t]\}_{t\geq 0}$ is a Markov chain because $a_i[t]$ depends only on $x_i[t-T_i]$, so $x_i[t+1]$ and $\mathbf{y}[t+1]$ are determined by $\mathbf{y}[t]$.

Foster's Criterion will be used to prove the main results in this paper. The basic idea is to find a Foster-Lyapunov function $W(\mathbf{y}[t])$ and a finite set S such that if $\mathbf{y}[t] \notin S$, then the drift $E[W(\mathbf{y}[t+1]) - W(\mathbf{y}[t])|\mathbf{y}[t]] < -\varepsilon$. The Foster-Lyapunov function that we will use in this paper is given by

$$W(\mathbf{y}[t]) = \frac{1}{2}\sum_{i}^{n}(x_i[t] - x_i^*)^2,$$

where the n is the number of users.

We will first show that $\{\mathbf{y}[t]\}$ is positive recurrent, which implies that it has a stationary distribution. Then, we can furthermore prove that as $K \to \infty$, the ith scaled queue length $(x_i/K^{1/m})$ tends to concentrate around the point $x_i^*/K^{1/m}$. Thus,

$$E[a_i] = \frac{\alpha_i}{\frac{(x_i[t])^m}{K}} \approx \frac{\alpha_i}{\frac{(x_i^*)^m}{K}}.$$

Therefore, we can approximate weighted m-fairness arbitrarily closely by choosing a large K. For a positive constant c and $\sigma < 1/m$ define

$$S_\sigma = \{\mathbf{y}[t] : \|\mathbf{x}[t] - \mathbf{x}^*\| \leq cK^\sigma \}. \tag{6}$$

Note that $\|\mathbf{x}[t] - \mathbf{x}^*\| \leq cK^\sigma$ implies that

$$\sum_i x_i[t-s] \leq \sum_i x_i^* + ncK^\sigma + nT\hat{\eta}, \quad \text{for all} \quad 0 \leq s \leq T.$$

Thus, S_σ is a finite set. Define

$$E[\Delta W_t(\mathbf{y})] := E\left[W(\mathbf{y}[t+1]) - W(\mathbf{y}[t])|\mathbf{y}[t]\right],$$

the following theorem shows $\mathbf{y}[t]$ is positive recurrent.

Theorem 2: There exist positive numbers $\sigma < 1/m$, c, δ^*, and ζ such that for large K

$$E[\Delta W_t(\mathbf{y})] \leq -\frac{\delta^*}{K^{\frac{1}{m}-\sigma}} \|\mathbf{x}-\mathbf{x}^*\| I_{\mathbf{y}\in S_\sigma^c} + \zeta I_{\mathbf{y}\in S_\sigma}, \tag{7}$$

where S_σ is defined as (6). Hence, the Markov chain $\{\mathbf{y}[t]\}$ is positive recurrent.

Proof: First of all, it is easy to see that if $\mathbf{y}[t] \in S_\sigma$, there exists $0 < \zeta < \infty$ such that $E[\Delta W_t(\mathbf{y})] < \zeta$. Now, consider $\mathbf{y}[t] \notin S_\sigma$, define the event χ_0^t such that

$$\chi_0^t := \left\{ \max_i \sum_{j=1}^{T} a_i[t-j] \leq A \right\},$$

and events χ_l^t for $l = 1, 2, \ldots$ such that

$$\chi_l^t := \left\{ \max_i \sum_{j=1}^{T} a_i[t-j] = A + l \right\}.$$

Then we can rewrite $E[\Delta W_t(\mathbf{y})]$ as follows:

$$E[\Delta W_t(\mathbf{y})] = \sum_{l=0}^{\infty} E[\Delta W_t(\mathbf{y}) | \chi_l^t] p(\chi_l^t).$$

For convenience, we also let $\{y\}^M$ denote $\min\{y, M\}$. Then, along the lines of the proof of Theorem 1 of [6], it can be shown that there exists $B_d > 0$, which is independent on K and $\mathbf{x}[t]$, such that

$$E[\Delta W_t(\mathbf{y})] \leq \sum_{i=1}^{n} \Delta x_i[t] \left(\left\{ \tfrac{\alpha_i K}{(x_i[t-T_i])^m} \right\}^M - \mu_i^* \right) + B_d \tag{8}$$

$$= \sum_{i=1}^{n} \Delta x_i[t] \left(\left\{ \tfrac{\alpha_i K}{(x_i[t-T_i])^m} \right\}^M - \left\{ \tfrac{\alpha_i K}{(x_i[t])^m} \right\}^M \right) \tag{9}$$

$$+ \sum_{i=1}^{n} \Delta x_i[t] \left(\left\{ \tfrac{\alpha_i K}{(x_i[t])^m} \right\}^M - \mu_i^* \right) + B_d, \tag{10}$$

where $\Delta x_i[t] = x_i[t] - x_i^*$. The difference between this analysis and the delay-free case is the additional term (9), which we refer to as $H(K)$. It has been shown in [6] that $G(K) := (10)$ is negative when $\|\mathbf{x}\|$ is large. Here, we do not know the sign of (9), but we can show that when K increases, the magnitude of the expression (9) increases much slower than the magnitude of the expression (10). So $|H(K)|/G(K) \to 0$ as $K \to \infty$, and when K is large enough, we will have $(9) + (10) = G(K) + H(K) < G(K)/2 < 0$, which implies $E[\Delta W_t(\mathbf{y})] < 0$.

To show this, we will show the following three facts. The first one is that there exists $\delta_d > 0$

such that for all events χ_l^t,

$$G(K) \le -\frac{\delta_d}{K^{\frac{1}{m}-\sigma}} \|\mathbf{x}[t] - \mathbf{x}^*\|. \tag{11}$$

Second, when χ_0^t happens, there exists $\delta_0 > 0$ such that

$$p(\chi_0^t)\,|H(K)| \le p(\chi_0^t)\frac{\delta_0}{K^{\xi}} \|\mathbf{x}[t] - \mathbf{x}^*\|. \tag{12}$$

The last one is that there exists $\delta_1 > 0$ such that

$$\sum_{l=1}^{\infty} p(\chi_l^t)\,|H(K)| \le \frac{\delta_1}{K^{\xi}} \|\mathbf{x}[t] - \mathbf{x}^*\|. \tag{13}$$

If all three inequalities — (11), (12) and (13) — hold and $\xi > \frac{1}{m} - \sigma$, we will have

$$\begin{aligned}
E[\Delta W_t(\mathbf{y})] &\le G(K) + p(\chi_0^t)H(K) + \sum_{l=1}^{\infty} p(\chi_l^t)H(K) \\
&\le -\left(\frac{\delta_d}{K^{\frac{1}{m}-\sigma}} - \frac{\delta_0 p(\chi_0^t) + \delta_1}{K^{\xi}}\right) \|\mathbf{x}[t] - \mathbf{x}^*\| \\
&\le -\left(\frac{\delta_d}{K^{\frac{1}{m}-\sigma}} - \frac{\delta_0 + \delta_1}{K^{\xi}}\right) \|\mathbf{x}[t] - \mathbf{x}^*\|.
\end{aligned}$$

Then, when $K > ((2\delta_0 + \delta_1)/\delta_d)^{1/(\xi+\sigma-1/m)}$, we have

$$\frac{\delta_d}{2K^{\frac{1}{m}-\sigma}} - \frac{\delta_0 + \delta_1}{K^{\xi}} > 0,$$

which implies

$$E[\Delta W_t(\mathbf{y})] \le -\left(\frac{\delta^*}{K^{\frac{1}{m}-\sigma}}\right) \|\mathbf{x}[t] - \mathbf{x}^*\|$$

and the theorem is proved with $\delta^* = \delta_d/2$.

Now we prove (11), (12) and (13). We will first show (11). The proof is similar to the proof of Theorem 1 of [6]. But here we consider a general m instead of $m = 1$. Define σ as follows:

$$\sigma = \begin{cases} \lfloor \frac{1}{m} \rfloor, & \text{if } m \le 1 \text{ and } \frac{1}{m} \text{ is not an integer;} \\ \frac{1}{m} - \frac{1}{2}, & \text{if } m \le 1 \text{ and } \frac{1}{m} \text{ is an integer;} \\ \frac{1}{2m}, & \text{if } m > 1. \end{cases}$$

From above definition, we have that

$$0 < \frac{1}{m} - \sigma < \min\{\sigma, 1\}.$$

Notice that we have

$$(x_i[t] - x_i^*)\left(\left\{\frac{\alpha_i K}{(x_i[t])^m}\right\}^M - \mu_i^*\right) \leq 0 \text{ for all } i.$$

Letting $i_0 = \arg\max_i |x_i[t] - x_i^*|$, then

$$G(K) \leq -|x_{i_0}[t] - x_{i_0}^*| \left|\left\{\frac{\alpha_{i_0} K}{(x_{i_0}[t])^m}\right\}^M - \mu_{i_0}^*\right| + B_d.$$

Now, if $\left\{\frac{\alpha_i K}{(x_{i_0}[t])^m}\right\}^M = M$, then from the definition of M, we have $M > 2\hat{\eta}$, so

$$\left|\left\{\frac{\alpha_{i_0} K}{(x_{i_0}[t])^m}\right\}^M - \mu_{i_0}^*\right| = M - \mu_{i_0}^* > \hat{\eta}.$$

Otherwise if $\left\{\frac{\alpha_i K}{(x_{i_0}[t])^m}\right\}^M < M$, then

$$\left\{\frac{\alpha_i K}{(x_{i_0}[t])^m}\right\}^M = \frac{\alpha_i K}{(x_{i_0}[t])^m}$$

and

$$\left|\left\{\frac{\alpha_{i_0} K}{(x_{i_0}[t])^m}\right\}^M - \mu_{i_0}^*\right| = \mu_{i_0}^* \left|\left(\frac{x_{i_0}^*}{x_{i_0}[t]}\right)^m - 1\right|.$$

Furthermore, because

$$x_{i_0}[t] = \begin{cases} x_{i_0}^* - |x_{i_0}[t] - x_{i_0}^*| \geq 0, & \text{if } x_{i_0}[t] - x_{i_0}^* \leq 0; \\ x_{i_0}^* + |x_{i_0}[t] - x_{i_0}^*| \geq 0, & \text{if } x_{i_0}[t] - x_{i_0}^* \geq 0, \end{cases}$$

and

$$\left|\left(\frac{x_{i_0}^*}{x_{i_0}^* - |x_{i_0}[t] - x_{i_0}^*|}\right)^m - 1\right| \geq \left|\left(\frac{x_{i_0}^*}{x_{i_0}^* + |x_{i_0}[t] - x_{i_0}^*|}\right)^m - 1\right|,$$

we have

$$\left|\left\{\frac{\alpha_{i_0} K}{(x_{i_0}[t])^m}\right\}^M - \mu_{i_0}^*\right| \geq \mu_{i_0}^* \left|\left(\frac{x_{i_0}^*}{x_{i_0}^* + |x_{i_0}[t] - x_{i_0}^*|}\right)^m - 1\right| \geq \mu_{i_0}^* \left|1 - \frac{1}{(1+\varepsilon)^m}\right|,$$

where

$$\varepsilon = \frac{c\mu_{i_0}^{*\,1/m}}{\sqrt{n}\alpha_{i_0}^{\,1/m}} K^{\sigma-\frac{1}{m}} > 0,$$

and the last inequality holds because $x_{i_0}^* = \left(\frac{\alpha_{i_0}K}{\mu_{i_0}^*}\right)^{1/m}$ and $cK^\sigma \leq \|\mathbf{x}[t]-\mathbf{x}^*\| \leq \sqrt{n}|x_{i_0}[t]-x_{i_0}^*|$. Furthermore, we know that

$$(1+\varepsilon)^m \geq 1+m\varepsilon.$$

Thus,

$$\left|\left\{\frac{\alpha_{i_0}K}{(x_{i_0}[t])^m}\right\}^M - \mu_{i_0}^*\right| \geq \mu_{i_0}^*\left|1-\frac{1}{1+m\varepsilon}\right| = \frac{\mu_{i_0}^* m\varepsilon}{1+m\varepsilon}.$$

It is easy to see that for large K, we will have $\frac{\mu_{i_0}^* m\varepsilon}{1+m\varepsilon} < \hat{\eta}$, which implies for large K,

$$\begin{aligned} G(K) \leq & \quad -|x_{i_0}[t]-x_{i_0}^*|\left(\frac{\mu_{i_0}^* m\varepsilon}{1+m\varepsilon} - \frac{B_d}{|x_{i_0}[t]-x_{i_0}^*|}\right) \\ \leq & \quad -|x_{i_0}[t]-x_{i_0}^*|\left(\frac{\mu_{i_0}^*}{\frac{\sqrt{n}}{mc}\left(\frac{\alpha_{i_0}}{\mu_{i_0}^*}\right)^{\frac{1}{m}} K^{\frac{1}{m}-\sigma}+1} - \frac{B_d}{\frac{c}{\sqrt{n}}K^\sigma}\right). \end{aligned}$$

Because $\frac{1}{m}-\sigma \leq \sigma$, by choosing sufficiently large c, we can find a positive constant $\hat{\delta}$ and $\hat{K}$ such that for any $K \geq \hat{K}$

$$(10) \leq -\frac{\hat{\delta}}{K^{\frac{1}{m}-\sigma}}|x_{i_0}[t]-x_{i_0}^*| \leq -\frac{\delta_d}{K^{\frac{1}{m}-\sigma}}\|\mathbf{x}[t]-\mathbf{x}^*\|,$$

where $\delta_d = \hat{\delta}/\sqrt{n}$.

Next, we consider (12). It is the case that the arrivals are upper bounded by A. We will show that as K increases, $\left|\left\{\frac{\alpha_i K}{(x_i[t-T_i])^m}\right\}^M - \left\{\frac{\alpha_i K}{(x_i[t])^m}\right\}^M\right|$ decreases.

We use Fig. 2 to prove our result. Suppose that $m>0$, then $\alpha_i K/y^m$ is convex and

$$f(y) = \{\alpha_i K/y^m\}^M$$

is as Fig. 2. Now suppose $c-a=d-b=A$ and $f(a)=M$. Then for any b such that $a<b$, we

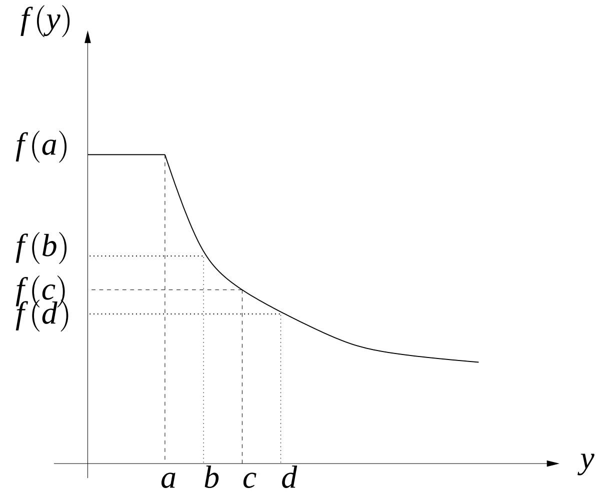

Fig. 2. $f(y) = \{\alpha_i K / y^m\}^M$

can see from the figure that $f(a) - f(c) > f(b) - f(d)$. It means that

$$\max_{x_1, x_2 : x_2 - x_1 \le A} f(x_1) - f(x_2) = f(a) - f(c).$$

Furthermore, from the assumption we have $A \ge T\hat{\eta}$, so $|x_i[t] - x_i[t-T]| \le A$ and

$$|f(x_i[t-T_i]) - f(x_i[t])| \le M - \frac{\alpha_i K}{\left(\left(\frac{\alpha_i K}{M}\right)^{\frac{1}{m}} + A\right)^m} = M - \frac{\alpha_i K}{\frac{\alpha_i K}{M}\left(1 + \left(\frac{A^m M}{\alpha_i K}\right)^{\frac{1}{m}}\right)^m}.$$

When K is large, $\left(\frac{A^m M}{\alpha_i K}\right)^{\frac{1}{m}}$ is small. Furthermore, for sufficient small ε, we have $(1+\varepsilon)^m \le 1 + 2m\varepsilon$. Then, we can conclude that for sufficient large K

$$|f(x_i[t-T_i]) - f(x_i[t])| \le M - \frac{M}{1 + 2m\left(\frac{A^m M}{\alpha_i K}\right)^{\frac{1}{m}}} < \frac{2mAM^{\frac{1+m}{m}}}{\alpha_i^{\frac{1}{m}}} \frac{1}{K^{\frac{1}{m}}}. \tag{14}$$

Let $\alpha_{\min} = \min_i \alpha_i$, then there exists $\delta_0 = 2n\alpha_{\min}^{-\frac{1}{m}} mAM^{\frac{1+m}{m}}$ and $\xi = \frac{1}{m}$ such that

$$|H(K)| \le \frac{\delta_0}{K^\xi} \|\mathbf{x}[t] - \mathbf{x}^*\|.$$

Finally, we consider the complement of χ_0^t, which is denoted as $\chi_0^{t\,c}$, and derive inequality (13). Now the arrivals are not upper bounded and can be arbitrarily large. But from assumption (3), the probabilities of these events is very small. So we can still obtain a upper bound for

$\sum_{i=1}^{n}(x_i[t]-x_i^*)\left|\left\{\frac{\alpha_i K}{x_i[t-T_i]}\right\}^M-\left\{\frac{\alpha_i K}{x_i[t]}\right\}^M\right|$. Now suppose χ_l^t occurs ($l \geq 1$), similar to inequality (14), we can get

$$\left|\left\{\frac{\alpha_i K}{x_i[t-T_i]}\right\}^M-\left\{\frac{\alpha_i K}{x_i[t]}\right\}^M\right| < 2\alpha_i^{-\frac{1}{m}} mM^{\frac{1+m}{m}}(A+l)\frac{1}{K^{\xi}}$$

and

$$\sum_{l=1}^{\infty} p(\chi_l^t)\,|H(K)| \leq \|\mathbf{x}[t]-\mathbf{x}^*\|\sum_{l=1}^{\infty} 2n\alpha_{\min}^{-\frac{1}{m}} mM^{\frac{1+m}{m}}\frac{A+l}{K^{\xi}}p(\chi_l^t).$$

Now we use assumption (3), which yields

$$\begin{aligned}\sum_{l=1}^{\infty} 2n\alpha_{\min}^{-\frac{1}{m}} mM^{\frac{1+m}{m}}\frac{1}{K^{\xi}}(A+l)p(\chi_l^t) &\leq \sum_{l=1}^{\infty} 2n\alpha_{\min}^{-\frac{1}{m}} mM^{\frac{1+m}{m}}\frac{1}{K^{\xi}}(A+l)\frac{\theta}{(A+l)^h}\\ &\leq 2n\alpha_{\min}^{-\frac{1}{m}} mM^{\frac{1+m}{m}}\frac{1}{K^{\xi}}(h-2)\frac{1}{A^{h-2}}=\frac{\delta_1}{K^{\xi}},\end{aligned}$$

where $\delta_1 = 2n\alpha_{\min}^{-\frac{1}{m}} mM^{\frac{1+m}{m}}(h-2)\frac{1}{A^{h-2}}$.

Thus, we have proved that inequalities (11), (12) and (13) hold. Also, it is easy to see that $\sigma < \xi$. Thus,Thus, when $K > 4(\delta_0+\delta_1)^2/\delta_d^2$, we have (7) with $\delta^* = \delta_d/2$. Furthermore, by invoking Foster's Criterion, we have that the Markov chain $\{\mathbf{y}[t]\}$ is positive recurrent. $\diamond$

Now, following an argument similar to that of Theorem 2 in [6], we can show the following result.

Theorem 3: There exists a positive constant $\bar{c}$, that depends on the mean achievable rate region, the algorithm parameters $\{\alpha_i\}$, and the moments of the channel and arrival process, such that

$$E\left[\|\mathbf{x}[\infty]-\mathbf{x}^*\|\right] \leq \bar{c}K^{\frac{1}{m}-\sigma} \text{ for large } K,$$

where $\mathbf{x}[\infty]$ is an informal notation for the steady state of $\mathbf{x}$ and $\|\cdot\|$ denotes the Euclidean distance in the $\Re^n$. $\diamond$

Now using the Markov inequality, Theorem 3 yields

$$P\left(\frac{1}{K^{\frac{1}{m}}}\left|x_i[\infty]-x_i^*\right| > \varepsilon\right) \leq \frac{\bar{c}}{\varepsilon K^{\sigma}},$$

which implies that $\frac{\mathbf{x}[\infty]}{K^{1/m}} \approx \frac{\mathbf{x}^*}{K^{1/m}}$ for large K and

$$E[a_i] = \frac{\alpha_i}{\frac{x_i^m}{K}} \approx \frac{\alpha_i}{\frac{x_i^{*m}}{K}}.$$

So $\mu[t]$ converges to μ^* for large t in a probabilistic sense and the network is weighted m-fair according to Theorem 1.

These results allow us to conclude that even in the presence of delays, the network will achieve weighted m-fairness.

From the above, we have shown that when K is large, the network is stable. It raises the concern that the network may not be stable for small K, creating the risk of operating the network at an unstable regime. But actually condition (7) is much stronger than what is necessary to prove the stability. Actually, for any $K > 0$, the Markov chain is positive recurrent. Define the $S_{\bar{X}}$:

$$S_{\bar{X}} = \left\{ \mathbf{y}[t] : \sum_i x_i[t] \le \bar{X} \right\}. \tag{15}$$

Clearly, $S_{\bar{X}}$ is a finite set.

Theorem 4: For any $K > 0$, there exists positive numbers ζ, $\bar{X}$ and δ such that

$$E[\Delta W_t(\mathbf{y})] \le -\delta \sum_{i=1}^{n} x_i[t] I_{\mathbf{y} \in S_{\bar{X}}^c} + \zeta I_{\mathbf{y} \in S_{\bar{X}}},$$

where $S_{\bar{X}}$ is defined as (15). Hence, the Markov chain $\{\mathbf{y}[t]\}$ is positive recurrent.

Proof: We omit the proof here because of lack of the space. Please refer Theorem 4 of [21] for the proof of $m = 1$. The case of general weighted-m fairness is similar. ■

IV. Conclusion

In this paper, we have shown that the algorithm (4) and (5) is stable even in the presence of heterogeneous delays and when K is large, the network will achieve weighted m-fairness. When delays are not negligible in some situations, our result reinforces the result that the combination of queue-length-based scheduling and congestion control is a good distributed fair resource allocation scheme.

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