1. (a) Find an example of a linear operator $T : V \rightarrow V$ on a Banach space $V$ which is injective but not surjective.

(b) Given an example of linear operators $T$ and $S$ such that $\|T \circ S\| < \|T\| \|S\|$. 

2. Let $H_1$ and $H_2$ be Hilbert spaces and $T \in B(H_1, H_2)$. Show that $\|T^* T\| = \|T\|^2$. 

3. (a) Consider a linear operator $A : X \rightarrow Y$. Assume that the domain of $A$ is $X$. If there is an operator $B : Y \rightarrow X$ with domain $Y$ s.t $AB = I_Y$ and $BA = I_X$, where $I_X$ and $I_Y$ are identity operators on $X$ and $Y$ respectively. Then show that $A$ has an inverse (i.e., $A$ is a bijection) and $B = A^{-1}$.

(b) Let $X$ be a Banach space, and $Q$ a linear bounded operator mapping $X$ into itself. Show that if the norm of $Q$ is less than 1, the operator $(I - Q)$ is invertible, and its inverse admits the infinite series representation 

$$ (I - Q)^{-1} = \sum_{n=0}^{\infty} Q^n $$

Note that we take $Q^0 = 1$.

[This is known as the Neumann series.]

*Hint:* Let $A_k = \sum_{n=0}^{k} A^n$. Show that $A_k$ is Cauchy sequence in $B(X, X)$. 

4. Let $X$ be a Banach space, and $\| \cdot \|_1$ and $\| \cdot \|_2$ be two different norms on it. Suppose that there exists a constant $\alpha$ such that $\|x\|_1 \leq \alpha \|x\|_2$ for all $x \in X$. Show that there exists another constant, $\beta$, such that $\|x\|_2 \leq \beta \|x\|_1$ for all $x \in X$.

*Hint:* Use the Banach Inverse Theorem on page 149 of the text. 

5. Let $X = L_p[0, 1]$, $1 < p < \infty$ and $Y = L_q[0, 1]$, $1/p + 1/q = 1$. Define $A$ by

$$ Ax = \int_0^1 K(t, s) x(s) ds. $$

Show that $A \in B(X, Y)$ if $\int_0^1 \int_0^1 |K(t, s)|^q dt ds < \infty$.

[This is problem 2 in page 166 in the book.] 

6. Let $X = L_p[0, 2]$, $1 < p < \infty$, $Y = L_q[0, 2]$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $A \in B(X, Y)$ be defined by

$$ A(x) = \int_0^2 K(t, s) x(s) ds $$

where

$$ \int_0^2 \int_0^2 |K(t, s)|^q dt ds < \infty $$

Find $A^*$, the adjoint of $A$. 