1. A linear transformation on a vector space $X$ may be continuous with respect to some norm on $X$, but discontinuous with respect to another norm on $X$. To illustrate this, let $X$ be the space $c_0$ of all sequences with only finitely many nonzero terms. This is a subspace of $l_1 \cap l_2$. Consider the linear transformation $T : c_0 \to \mathbb{R}$ given by

$$T((x_n)_{n \in \mathbb{N}}) = x_1 + x_2 + x_3 + \ldots, \quad (x_n)_{n \in \mathbb{N}} \in c_0$$

(a) Let $c_0$ be equipped with the $l_1$ norm. Prove that $T$ is a bounded linear functional from $(c_0, \| \cdot \|_1)$ to $\mathbb{R}$.

(b) Let $c_0$ be equipped with the $l_2$ norm. Prove that $T$ is not a bounded linear functional from $(c_0, \| \cdot \|_2)$ to $\mathbb{R}$.

Hint: Consider the sequences $(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{m}, 0, \ldots), m \in \mathbb{N}$.

2. (a) Let $B$ be a Banach space and let $f : B \to \mathbb{R}$ be a bounded linear functional. Show that the set $A = \{ x \in B : f(x) = \| f \| \}$ is closed and convex.

(b) Define $f : l_1 \to \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty} (1 - 1/n)x_n$. Show the following:

(i) $\| f \| = 1$.

(ii) The closed convex set $\{ x \in l_1 : f(x) = 1 \}$ has no closest point to 0.

3. (a) Let $H$ to be a Hilbert space and let $f$ be a bounded linear functional. Show that there exists $y \in H$ such that $\| y \| = 1$ and $f(y) = \| f \|$.

(b) Let $x = \{ x_n \}_{n \in \mathbb{N}} \in l_\infty$ and let $T_x : l_1 \to \mathbb{R}$ be defined by $T_x(y) = \sum_{n=1}^{\infty} x_n y_n$. What condition on $x$ is needed so that there exists $y \in l_1$ such that $\| y \|_1 = 1$ and $|T_x(y)| = \| T_x \|$?

Hint: Use Hölder’s inequality and first evaluate $\| T_x \|$.

4. Let

$$c_0 = \left\{ x = \{ x_n \}_{n \in \mathbb{N}} \in l_\infty : \lim_{n \to \infty} x_n = 0 \right\}.$$ 

Then, $c_0$ is a Banach space with respect to $\| \cdot \|_\infty$. Find an isometric isomorphism between $(c_0)^*$ and $l_1$. Why does the same argument not work for $(l_\infty)^*$ and $l_1$?

5. (a) Consider a normed space $X$. Show that

$$\| x \| = \sup_{f \in X^*, \| f \| = 1} |f(x)|$$

and the sup is attained.
(b) If \( x \) in a normed space \( X \) is such that \( |f(x)| \leq c \) for all \( f \in X^* \) of norm at most 1, show that \( ||x|| \leq c \).

6. Let \( X \) be a normed vector space and \( Z \) its subspace. Prove that if \( y \in X \) has distance \( d \) from \( Z \), then there exists a linear functional \( \Lambda : X \rightarrow \mathbb{R} \) such that

\[
||\Lambda|| \leq 1, \quad \Lambda(y) = d, \quad \text{and} \quad \Lambda(z) = 0, \forall z \in Z.
\]

(The distance referred is \( d = \inf_{z \in Z} ||z - y|| \))

7. Let \( g_1, g_2, \ldots, g_n \) be linearly independent linear functionals on a vector space \( X \). Let \( f \) be another linear functional on \( X \) such that for every \( x \in X \) satisfying \( g_i(x) = 0, i = 1, 2, \ldots, n \), we have \( f(x) = 0 \). Show that there exist constants \( \lambda_1, \lambda_2, \ldots, \lambda_n \) such that

\[
f = \sum_{i=1}^{n} \lambda_i g_i.
\]

Hint: Let \( S := \{ s_1 = g_1(x), s_2 = g_2(x), \ldots, s_n = g_n(x) : x \in X \} \). Show that \( S \) is a subspace of \( \mathbb{R}^n \) and then use the Hahn-Banach Theorem (extension form).