1. Let $H$ be a Hilbert space and $A \subset H$.
   (a) Show that $A^\perp$ is a closed subspace of $H$.
   (b) Show that $A^{\perp\perp} = \text{span}(A)$, i.e., $A^{\perp\perp}$ is the smallest closed subspace containing $A$.
       Also note that this means that $A = A^{\perp\perp}$ if and only if $A$ is a closed subspace of $H$.
   (c) Suppose that $\text{span}(A) = H$. Show that if there exists $u, v \in H$ such that $(x|u) = (x|v)$ for all $x \in A$ then $u = v$.

2. In $l_2$ find an example of a subspace $A \subset l_2$ where $A \oplus A^\perp \neq l_2$. In other words show that the assumption that the subspace is closed is necessary in the Decomposition theorem.

3. (a) Let $X$ denote the space of square integrable functions on the closed finite interval $[-1, 2]$. Let $(\cdot, \cdot)$ be a mapping from $X \times X$ onto $\mathbb{R}$, defined by
      \[ (x, y) = \int_{-1}^{2} t^2 x(t) y(t) \, dt \]
      Determine whether $(\cdot, \cdot)$ is an inner product on $X$. If yes, is $X$ a Hilbert space?
   (b) We seek an affine function $m(t) = a + bt$ that minimizes the integral
      \[ F(m) = \int_{-1}^{2} t^2 [t^3 - m(t)]^2 \, dt . \]
      (i) Formulate this as a projection problem in a Hilbert space $H$, by clearly identifying both $H$ and the subspace $M$.
      (ii) Obtain the solution. Is it unique?
      (iii) What is the minimum value of $F$?

4. Obtain the solution to the problem of minimizing the functional $\int_{1}^{2} t x^2(t) \, dt$ subject to two constraints:
      \[ \int_{1}^{2} x(t) t^{1/3} \, dt = 1 \quad \text{and} \quad \int_{1}^{2} x(t) t^{2/3} \, dt = -1 \]
      Is the solution unique?
      \textit{Hint:} Let $y(t) = x(t)\sqrt{t}$. Formulate and solve the problem in terms of $y(t)$.

5. (a) Show that $(x, y) = x^T Q y$ is an inner product on $\mathbb{R}^n$ when $Q$ is symmetric and positive definite.
(b) Using the Projection Theorem, solve the finite-dimensional optimization problem:

\[
\text{minimize } x^TQx \quad \text{subject to } Ax = b
\]

where \(x\) is an \(n\)-vector, \(Q\) a positive-definite (symmetric) matrix, \(A\) an \(m \times n\) matrix (with \(m < n\)), and \(b\) an \(m\)-vector.

[You should be able to obtain a closed-form expression in terms of \(A, Q\) and \(b\).]

6. Consider the problem of finding the vector \(x\) of minimum norm satisfying the inequality constraints:

\[(x, y_i) \geq c_i, \quad i = 1, 2, \ldots, n\]

where the \(y_i\)'s are linearly independent, and the \(c_i\)'s are given constants.

(i) Show that this optimization problem admits a unique solution.

(ii) Show that a necessary and sufficient condition for a vector \(x = \sum_{i=1}^{n} a_i y_i\) to constitute a solution to this problem is that the vector \(a\) with components \(a_i\) satisfy \(G^T a \geq c; \quad a \geq 0,\) and that \(a_i = 0\) if \((x, y_i) > c_i\). Here \(G\) is the Gram matrix of \(\{y_1, y_2, \ldots, y_n\}\).

7. The following theorem is valid in a Hilbert space \(H\). If \(K\) is a closed convex set in \(H\) and \(x \in H, x \notin K\); there is a unique vector \(k_0 \in K\) such that \(||x - k_0|| \leq ||x - k||\) for all \(k \in K\). Show that this theorem does not apply in arbitrary Banach space.

\textit{Hint:} Think of an example in \(\mathbb{R}^2\) with the \(l_\infty\) norm.