The following property of real numbers may be assumed to solve the problems.

**Continuum Property:**

Let $S$ be a bounded set of real numbers, i.e., $\exists M < \infty$ s.t. $|x| \leq M \ \forall \ x \in S$. Then, $\exists \ M$ s.t.

- $x \leq M \ \forall \ x \in S$ and
- $\forall \epsilon > 0, \ \exists \ y \in S$ s.t. $y \geq M - \epsilon$.

In other words, $M$ is the least upper bound or supremum of $S$. Similarly, there exists a greatest lower bound, or infimum of $S$.

**Note:** The problems in this homework are designed to refresh your knowledge of undergraduate real analysis. The solutions can probably found in standard books or by Googling for them since these are basic real analysis facts. However, you are not allowed to simply copy the solutions from some website or book, please write the answers in your own words.

**Problems**

1. Let $x_n, \ n = 1, 2, \ldots$ be an infinite sequence of real numbers. Prove that it has a subsequence that is monotone, i.e., a subsequence that is either non-decreasing or non-increasing.
   
   **Note:** A subsequence is an infinite subset $x_{n_1}, x_{n_2}, \ldots$ such that $n_1 < n_2 < \cdots$.

2. Show that every bounded sequence of real numbers, $x_n, \ n = 1, 2, \ldots$ has a convergent subsequence (this is called the Bolzano-Weierstrass Theorem).
   
   **Note:** A sequence $\{x_n\}$ is bounded if $\exists M < \infty$ s.t. $|x_n| \leq M \ \forall \ n$. Note that $\{x_n\}$ is another notation for an infinite sequence.

3. A sequence of real numbers $\{x_n\}$ is said to be a Cauchy sequence if given $\epsilon > 0, \ \exists \ N_\epsilon$ s.t. $|x_n - x_m| \leq \epsilon \ \forall \ n, m > N_\epsilon$. Show that every Cauchy sequence is bounded.

4. Show that if a Cauchy sequence of real numbers has a convergent subsequence, then the sequence itself must converge.

5. Prove that every Cauchy sequence of real numbers is convergent.