9. Dual spaces

In a Hilbert space, the projection problem can be solved by invoking the orthogonality principle:

\[(x - \hat{x} | y) = 0 \quad \forall \ y \in M\]

But what if the vector space does not have an inner product? The study of the so-called dual space allows to extend the projection theorem partially to Banach spaces without an inner product.

Let \( x \) be a Banach space, and let \( f \) be a linear functional, i.e., \( f : x \to \mathbb{R} \). \( f \) is said to be bounded if

\[ |f(x)| \leq M \|x\| + x \in x.\]
Lemma:

(a) If a linear functional $f$ is continuous at one point $x_0$, then it is continuous at all $x \in X$.

(b) $f$ is continuous if and only if it is bounded.

Proof:

(a) Suppose $x_n \to x$ as $n \to \infty$.

$f(x_n) = f(x_n - x + x_0 + x_0) = f(x_n - x + x_0) + f(x_0)$

$f$ is linear.

Again, $f$ is linear.

$f(x_n - x + x_0) \to f(x) - f(x_0) + f(x) \to 0$ as $x_n \to x$

since $f$ is continuous at $x_0$

$f(x) \to f(x)$ as $x_n \to x$

(b) $(\Rightarrow)$ Suppose $f$ is continuous at $0$.

Then $f$ is continuous.
\[ |f(x)| \leq 1 \quad \Rightarrow \quad \|x\| \leq \delta \]

Consider any \( y \in X \). Then \( \|\frac{\delta y}{\|y\|}\| < \delta \)

\[
f \left( \frac{\delta y}{\|y\|} \right) = \frac{\delta}{\|y\|} f(y) \quad \text{by linearity}
\]

\[
\Rightarrow \quad \|f(y)\| = \frac{\|y\|}{\delta} \left| \frac{f(\delta y)}{\|y\|} \right|
\]

\[
\leq \frac{\|y\|}{\delta} \quad \Rightarrow \quad \text{bounded with} \quad M = \frac{1}{\delta}
\]

(\(\Leftarrow\)) Suppose \( f \) is bounded

Let \( x_n \to 0 \)

\[
|f(x_n)| \leq M \|x_n\| \to 0 = f(0)
\]

\[ f(0) = f(x) \]

\[ f(x) = \|x\| \quad \text{linear.} \]

The space of all bounded linear functionals of \( X \) is denoted \( X^* \)

\[ (f + g)(x) = f(x) + g(x) \]

\[ \alpha f(x) = \alpha f(x) \]

\[ \text{a vector space.} \]
Elements of $X^*$ are denoted as $f, g, h$ etc. or by $x^*$. The value that a functional takes at a vector $x \in X$ is denoted by $f(x)$ or $x^*(x)$ or $\langle x, x^* \rangle$. This last notation is suggestive of an inner product although it is not an inner product.

Norm in $X^*$:

$$
\|f\| = \inf \left\{ M : |f(x)| \leq M \|x\| \quad x \in X \right\}
$$

$$
= \sup_{x \to 0} \frac{|f(x)|}{\|x\|}
$$

$$
= \sup_{\|x\| \leq 1} |f(x)|
$$

$$
= \sup_{\|x\| = 1} |f(x)|
$$

It is easy to verify these equality using the fact that $f$ is linear and bounded.
\(-X^*\) is called the normal dual or simply the dual space of \(X\).

**Theorem:** \(X^*\) is a Banach space.

**Proof:** It is easy to verify that \(X^*\) is a vector space and \(\| f \|\) is a norm. We only need to verify completeness.

**Step 1:** Let \(\{x_n^*\} \in X^*\) be Cauchy, i.e.

\[\|x_n^* - x_m^*\| \leq \varepsilon \quad \forall n, m \geq N_0\]

Then,

\[|x_n^*(x) - x_m^*(x)| = \|x_n^* - x_m^*\|(x)\]

\[\leq \|x_n^* - x_n^*\| \|x\|\]

\[\leq \varepsilon \|x\| \quad \forall n, m \geq N_0 \quad \forall x\]

\(\Rightarrow\) the real numbers \(\{x_n^*(x)\}\) is Cauchy and they converge to some number \(x^*(x)\).
\[ \alpha^*(\alpha x + \beta y) = \lim_{n \to \infty} \alpha^*(\alpha x_n + \beta y_n) \]

\[ = \lim_{n \to \infty} \alpha x_n^* + \beta y_n^* \]

\[ = \alpha x^*(x) + \beta y^*(y), \]

thus linearity is verified. To verify boundedness, note that

\[ |x^*(x) - x_n^*(x)| \leq |x^*(x) - x_m^*(x)| + |x_m^*(x) - x_n^*(x)| \]

\[ \leq |x^*(x) - x_m^*(x)| + \|x_m^* - x_n^*\| \times \|x\| \]

\[ \leq |x^*(x) - x_m^*(x)| + \varepsilon \times \|x\| \]

\[ \forall n,m \geq N \in \mathbb{N} \]

since \( \{x_n^*\} \) is Cauchy

Letting \( m \to \infty \), \( n \geq N \in \mathbb{N} \)

\[ |x^*(x) - x_n^*(x)| \leq \varepsilon \|x\| \]

\[ \Rightarrow |x^*(x)| \leq |x_n^*(x)| + \varepsilon \|x\| \]

\[ \leq (M_n + \varepsilon) \|x\| \]

(Where \( M_n \) is the \( \|x\| \) of \( x_n^* \))

\[ \Rightarrow x^* \text{ is bounded.} \]
Step 3: We have to verify that $x_n^* \to x^*$ in the dual norm. We have already shown that

$$\|x^*(x) - x_n^*(x)\| \leq \varepsilon \|x\| + n \geq N\varepsilon$$

$$\Rightarrow \|x^* - x_n^*\| \leq \varepsilon \quad \forall \ v \geq N\varepsilon$$

$$\Rightarrow x_n^* \to x^* \text{ in the dual norm. }$$

Next, we will study the duals of some common Banach spaces.
Theorem (Riesz-Frechet): The dual of a Hilbert space is itself, i.e., let $H$ be a Hilbert space, and let $f$ be a linear functional on $H$. Then

$$f(x) = (x,y)$$

for a unique $y \in H$, and

$$\|f\| = \|y\|.$$ 

Further, for every $y$, $f(x)$ is a bounded linear functional.

Proof: Given $y$, consider $f(x) = (x,y)$. Clearly this is linear and

$$|f(x)| \leq \|y\| \|x\| \quad (C-S)$$

$\Rightarrow f$ is bounded.

Now, we want to show that every linear functional can be represented
this way. If \( f(x) = 0 \) for some \( x \), then we can take \( y = 0 \).

Next, consider the case \( f(x) \neq 0 \) for at least one \( x \). Define

\[
N = \{ x : f(x) = 0 \}
\]

This is a closed subspace of \( H \). Consider by the continuity of \( f \), there exists \( z \in N^\perp \) s.t. \( z \neq 0 \) s.t. \( f(z) = 1 \). We can assume \( f(z) = 1 \), by appropriately scaling \( z \) since \( f \) is linear.

\[
f(x - f(x) z) = f(x) - f(x) f(z)
\]

\[
= 0
\]

\[
\Rightarrow x - f(x) z \in N
\]

\[
\Rightarrow x - f(x) z \perp z \text{ s.t. } z \in N^\perp
\]

\[
\Rightarrow (x|z) = f(x) (z|z) = f(x) \frac{z}{\|z\|^2}
\]

or \( f(x) = (x|\frac{z}{\|z\|^2}) \)
We have to finally show that \( \mathfrak{g} \) is unique. Suppose

\[ f(x) = (x | b_1) = (x | b_2) \]

\[ \Rightarrow (x | b_1 - b_2) = 0 \quad \forall x \]

\[ \Rightarrow b_1 - b_2 = 0 \quad \Rightarrow b_1 = b_2 \]

Thus, the dual of \( l_2 \) is \( l_2^* \).

The dual of \( l_2 \) is \( l_2^* \).

Dual of \( l_p \), \( 0 < p < \infty \) is \( l_q \),

where \( \frac{1}{p} + \frac{1}{q} = 1 \)

(Note: the above statement does not state that the dual of \( l_0 \) is \( l_\infty \).

In fact, this is not true.)

Theorem: Let \( x \in l_p \), \( 0 < p < \infty \).
(i) Then every bounded linear functional \( f \) on \( l^p \) can be written uniquely in the form

\[
f(x) = \sum_{i=1}^{\infty} x_i y_i,
\]

where \( y \in l^q \), with \( \frac{1}{p} + \frac{1}{q} = 1 \).

Further, \( \|f\| = \|y\|_q \).

(ii) \( f(x) = \sum_{i=1}^{\infty} x_i y_i \) defines an \( f \in l^p \) if \( y \in l^q \).

**Proof:**

(i) Suppose \( f \) is a bounded, linear functional. Define \( e_i = (0, 0, \ldots, 1, 0, 0, \ldots) \) in position \( i \).

Then

\[
f(x) = f\left( \sum_{i=1}^{\infty} x_i e_i \right)
\]

\[
= \sum_{i=1}^{\infty} x_i f(e_i) = y_i
\]

\[
= \sum_{i=1}^{\infty} x_i y_i.
\]
We have to show that \( y \in \mathbb{R}^q \).

Consider

\[
\sum_{i=1}^{n} |y_i|^q
\]

\[
= \sum_{i=1}^{n} \left( |y_i| \frac{q}{p} \right)^p \quad \text{(assume } 1 < p < \infty) \]

Define \( x_i = \begin{cases} |y_i|^{\frac{q}{p}} & \text{sgn } (y_i), \quad i \in \mathbb{N} \\ 0 & \text{else} \end{cases} \)

\[
f(x) = \sum_{i=1}^{n} x_i \cdot \text{sgn}(y_i) \cdot |y_i|^\frac{q}{p}
\]

\[
= \sum_{i=1}^{n} |y_i|^\frac{q}{p} + 1
\]

\[
= \sum_{i=1}^{n} |y_i|^\frac{q}{p + 1} \quad \text{(since} \quad \frac{q}{p + 1} = \frac{q}{p} \left( \frac{p + 1}{p} = q \right) \text{)}
\]

Also,

\[
|f(x)| \leq \| x \|_p \| f \|_p \,
\]

\[
= \left( \sum_{i=1}^{n} |y_i|^q \right)^\frac{1}{q} \| f \|_p
\]
Thus,
\[
\left( \sum_{i=1}^{N} \left| y_i \right|^q \right)^{\frac{1}{q'}} \leq \| f \|_q
\]

\[
\Rightarrow \left( \sum_{i=1}^{N} \left| y_i \right|^q \right)^{\frac{1}{q'}} \leq \| f \|_q
\]

since this is true for \( q' > q \),

\[\| y \|_q \leq \| f \|_q\]

The proof for \( p=1 \) is similar.

*Proof of \( \mathbb{L}_p \{0,1\} \leq p < \infty \) is similar.*

\[\frac{1}{p} + \frac{1}{q'} = 1\]

The dual of \( \mathbb{L}_p \) is \( \mathbb{L}_q \) in the same sense as before: there is a one-to-one correspondence between functionals on \( \mathbb{L}_p \) and \( \mathbb{L}_q \), i.e.,
If $u$ is a bounded linear functional on $L^p$ if it can be written as

$$ f(x) = \int \alpha(t) y(t) \, dt, $$

where $y \in L^q[0,1]$, and $\|f\| = \|y\|$. 
