2. Normed Space

- A vector space \( X \) over a field \( F (\mathbb{R} \text{ or } \mathbb{C}) \) is a set with two operations \( + \) and \( \cdot \) satisfying the following properties: \( \forall x, y, z \in X, \alpha, \beta \in F \)

- \( (x + y) + z = x + (y + z) \)
- \( x + y = y + x \)
- \( \exists 0 \in X \text{ s.t. } 0 + x = x \)
- \( \exists -x \text{ s.t. } x + (-x) = 0. \)

(Note: \( x + (-y) \) denoted by \( x - y \).)

- \( 1 \cdot x = x \)
- \( \alpha (\beta x) = (\alpha \beta) x \)
- \( (\alpha + \beta) x = \alpha x + \beta x \)
- \( \alpha (x + y) = \alpha x + \alpha y \)

Elements of \( X \) are called vectors, elements of \( F \) are called scalars.

Examples:

(i) \( \mathbb{R}^n \)

(ii) Infinite sequences with each element
belong to $F$:
\[ x = (x_1, x_2, x_3, \ldots) \]

(iii) Infinite sequences with a finite number of non-zero elements:
\[ x = (x_1, x_2, x_3, \ldots, x_n, 0, \ldots) \]

(iv) Space of continuous functions over $[0, T]$.

**Normed space:** A normed space $X$ is a vector space endowed with a norm $\| \cdot \|$. A norm has to satisfy the following properties:

(i) $\| x \| = 0$ iff $x = 0$

(ii) $\| a x \| = |a| \| x \|$

(iii) $\| x + y \| \leq \| x \| + \| y \|$ (triangle inequality)

**Examples:**

(i) $C[0,T]$: The space of continuous functions in $[0, T]$, with

\[ \| x \| = \max_{t \in [0, T]} |x(t)| \]

It is easy to verify that the
properties of a norm are satisfied.

\( C'[0,1] \): The space of continuously differentiable functions with

\[
\| x \| = \max_{t \in [0,1]} | x(t) | + \max_{t \in (0,1]} | x'(t) |
\]

Again, easy to verify that \( \| \cdot \| \) is a norm.

\( l_p \) space, \( 1 \leq p \leq \infty \):

\( 1 \leq p < \infty \),

\[
l_p = \left\{ (x_1, x_2, \ldots) : x_i \in \mathbb{F}, \quad \| x \|_p = \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} \leq \infty \right\}
\]

\( l_\infty = \left\{ (x_1, x_2, \ldots) : x_i \in \mathbb{F}, \quad \| x \|_\infty = \sup |x_i| < \infty \right\}
\]

here, \( \sup \) means the smallest upper bound since we are considering infinite sequences \( x \) and hence, a max may not exist.

Example: \( x = (0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \ldots ) \in l_p \)

with \( \| x \| = 1 \)
The proof that the norm associated with $l^p$ spaces is indeed a norm is not trivial. The first two properties of a norm can be shown easily, but the triangle inequality relies on Minkowski's inequality, which uses Holder's inequality, which uses the generalized AM-GM inequality.

**AM-GM inequality:** (Arithmetic mean is greater than or equal to the geometric mean)

If $a, b > 0$, then $\sqrt{ab} \leq \frac{a+b}{2}$

**Proof:** Square both sides,

$$(a+b)^2 \geq 4ab$$

Then $(a+b) \geq 2 \sqrt{ab}$

**Generalized AM-GM (a.k.a. Young's) inequality:**

$0 \leq \alpha \leq 1$, $a, b > 0$. Then

$$a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b$$

(Equality if $\alpha = 0$ or 1, or $a = b$)

**Proof:** Take log on both sides:

$$\alpha \log a + (1-\alpha) \log b \leq \log (\alpha a + (1-\alpha)b)$$
But this is the definition of strict concavity of log is strictly concave.

**Hölder's inequality:**

\[
\sum_{i=1}^{\infty} |x_i y_i| \leq \|x\|_p \|y\|_q
\]

if \( \frac{1}{p} + \frac{1}{q} = 1 \), \( 1 \leq p < \infty \). Equality holds if

\[
\left( \frac{|x_i|}{\|x\|_p} \right) = \left( \frac{|y_i|}{\|y\|_q} \right) \quad \forall i
\]

**Proof:** We will prove the result for \( 1 < p < \infty \). The cases \( p = 1 \) or \( \infty \) are easy. Rewrite the inequality as

\[
\sum_{i=1}^{\infty} \frac{|x_i|}{\|x\|_p} \frac{|y_i|}{\|y\|_q} \leq 1
\]

By **Young inequality**

\[
\frac{|x_i|}{\|x\|_p} \frac{|y_i|}{\|y\|_q} \leq \frac{|x_i|^p}{p \|x\|_p^p} + \frac{1}{q} \frac{|y_i|^q}{\|y\|_q^q}
\]

\[
\sum_{i} \frac{|x_i| |y_i|}{\|x\|_p \|y\|_q} \leq \frac{1}{p} + \frac{1}{q} = 1
\]
Equality \quad \frac{|x|_p}{\|x\|_p} = \frac{|y|_q}{\|y\|_q}

Raising to the power of \frac{1}{pq} gives the desired result.

The special case of \( p=q=2 \) is called the Cauchy–Schwarz inequality.

Minkowski's inequality

For \( 1 \leq p \leq \infty \), \( \|x+y\|_p \leq \|x\|_p + \|y\|_p \).

(Note: This establishes that \( \| \cdot \|_p \) is a norm by proving the triangle inequality since the other two conditions for a norm are easy to prove.)

Proof:

\[
\sum_i |x_i + y_i|^p = \sum_i |x_i + y_i|^{p-1} |x_i + y_i| \\
\leq \sum_i |x_i + y_i|^{p-1} |x_i| + \sum_i |x_i + y_i|^{p-1} |y_i| \\
\leq \left( \sum_i |x_i + y_i|^{q-1} \right)^{\frac{1}{q}} \|x\|_p + \left( \sum_i |x_i + y_i|^{q-1} \right)^{\frac{1}{q}} \|y\|_p
\]
\[
\begin{align*}
\left( \sum_i |x_i + y_i|^p \right)^\frac{1}{q} & \leq \left( \|x\|_p + \|y\|_q \right) \\
& = \left( \frac{p}{q} \right)^\frac{1}{q} \cdot \left( \|x\|_p + \|y\|_q \right) \\
& = \|x + y\|_p \\
& \Rightarrow \left( \|x + y\|_p \right)^{\frac{1}{q}} \leq \|x\|_p + \|y\|_q \\
& \Rightarrow \|x + y\|_p \leq \|x\|_p + \|y\|_q
\end{align*}
\]

\[\square\]

\text{\underline{$l^p$ spaces.}} \quad \text{The space of functions} \quad \text{in} \quad [0, T] \quad \text{for which} \quad \|x\|_p = \left( \int_0^T |x(t)|^p \, dt \right)^{\frac{1}{p}} \quad \text{exists is called the} \quad l^p[0, T] \quad \text{space,} \quad 1 \leq p < \infty.

\text{For} \quad p = \infty, \quad \text{the same definition holds} \quad \|x\|_\infty = \sup_{t \in [0, T]} |x(t)|.

\text{As in the case of $l^p$ space, one uses an integral version of Holder's} \quad \text{and Minkowski inequalities to prove}
that $\|x\|_p$ is indeed a norm, but we will not do so here.

Note: $\int_0^T |x(t)|^p \, dt$ is not affected if we change $x(t)$ in a set of measure zero in $[0, T]$. Functions that differ in a set of measure zero in $L_p$ are treated as the same element of $L_p$. Similarly, the essential supremum of $x$ is the same as a supremum but ignores what happens in sets of measure zero.

$$\|x\|_\infty = \text{ess sup } |x(t)|$$

$$= \inf \left\{ \sup_{x(t) = y(t), t \in [0, T]} |y(t)| \right\} \text{ a.e.}$$

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a.e. means almost everywhere.
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The definition of $L_p[0, T]$ can be extended to functions \( \{x(t), a \leq t \leq b\} \) in the natural manner, in which case the space is called $L_p[a, b]$. 
Dimension of a vector space $X$

(This discussion does not require a normed space)

- Let $\mathcal{B} = \{a_1, a_2, \ldots, a_n\}$, where $a_i \in X$. The span of $\mathcal{B}$, $\text{span}(\mathcal{B})$, is defined as:

$$\{ x \in X : x = \sum_{i=1}^{n} \alpha_i a_i \text{ for } \alpha_i \in \mathbb{F} \}$$

- $\mathcal{B}$ is said to span $X$ if $X = \text{span}(\mathcal{B})$.

- $X$ is said to be a linear combination of $\mathcal{B}$ if

$$x = \sum_{i=1}^{n} \alpha_i a_i \text{ for } \alpha_i \in \mathbb{F}$$

- $\mathcal{B}$ is said to be linearly independent if no $a_i \in \mathcal{B}$ can be written as a linear combo of $\mathcal{B} \setminus a_i$. 
A linearly independent set $A$ is said to be a basis for $X$ if $X = \text{span}(A)$.

- $X$ is said to be finite-dimensional if $X = \text{span}(A)$ for some finite set $A$. $X$ is infinite-dimensional if there exists no such finite set $A$.

**Theorem:** If $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$ are two bases for $X$, then $m = n$, i.e., two bases for $X$ have the same number of elements. $n$ is called the dimension of $X$.

**Proof:** Suppose $m < n$. Since $B$ is a basis for $X$, $\text{span}(B) = X$. Further, $\text{span} \{a_1, b_1, b_2, \ldots, b_m\} = X$. Note that $\text{span} \{a_1, b_1, b_2, \ldots, b_m\} = X$. Note that $a_i$ can be written as $\sum_{i=1}^{m} c_i b_i$.

$a_i$ can be written as $\sum_{i=1}^{m} c_i b_i$.
at least one \( a_i \neq 0 \). Assume \( a_i \neq 0 \).

\[ a_1 = a_1 b_1 + \sum_{i=2}^{m} \frac{a_i}{a_1} b_i \]

\[ b_1 = \frac{1}{a_1} a_1 - \frac{1}{a_1} \sum_{i=2}^{m} \frac{a_i}{a_1} b_i \]

Thus, \( b_i \) can be expressed as a linear combo of \( \{a_1, b_2, \ldots, b_n\} \).

\[ X = \text{span} \{a_1, b_2, \ldots, b_m\} \]

Next, use the fact \( a_2 \) can be written as a linear combo of \( a_1, b_2, \ldots, b_n \) to show that

\[ X = \text{span} \{a_1, a_2, b_3, \ldots, b_n\} \]

Continuing,

\[ X = \text{span} \{a_1, a_2, \ldots, a_m\} \]

\[ \{a_1, \ldots, a_m\} \text{ is not linearly independent.} \]

\[ \Rightarrow \text{ contradiction } \Rightarrow n = m. \]