12. Min norm problems in Banach spaces

Recall the result in Hilbert spaces:

\[
\min_{m \in M} \| x - m \| \quad (M: \text{ subspace})
\]

\( m_0 \) solves the above problem iff \( \langle x - m, m \rangle = 0 \ \forall \ m \in M \).

This is impossible to extend to Banach spaces since we don't have the concept of an inner product in general. A generalization to Banach spaces if we rewrite the above condition differently.

Claim: \( \langle x - m, m \rangle = 0 \ \forall \ m \in M \)

iff \( \exists y \neq 0 \text{ s.t. } \langle m, y \rangle = 0 \ \forall \ m \in M \)

and \( \langle x - m_0, y \rangle = \| x - m_0 \| \| y \| \)

Proof:

(\( \Rightarrow \)): Let \( y = \frac{x - m_0}{\| x - m_0 \|} \)

Then \( \langle m, y \rangle = 0 \ \forall \ m \in M \).

Furthermore, \( \langle x - m_0, y \rangle = \langle x - m_0, \frac{x - m_0}{\| x - m_0 \|} \rangle = \| x - m_0 \| \| y \| \)
\((\leq)\) \quad (x-m_0 \mid m) \leq \|x-m_0\| \|m\|

\[= \frac{(x-m_0 \mid y)}{\|y\|} \quad \|m\| \]

\[= 0 \quad \forall m.
\]

So the Hilbert space result can be rewritten as follows:

\[m_0 \text{ solves } \min_{m \in M} \|x-m\| \]

iff \( \exists y \in X \) s.t. \( y \in M^\perp \)

and \( (x-m_0 \mid y) = \|x-m_0\| \|y\| \).

Definition of \( M^\perp \) in Banach space

\[M^\perp = \{ x^* \in X^* : \langle x, x^* \rangle = 0 \quad \forall x \in M \}\]

Thus, \( M^\perp \subseteq X^* \). \( M^\perp \) is called the orthogonal complement of \( M \).

If \( M \subseteq X^* \), then

\[M^\perp = \{ x^{**} \in X^{**} : \langle x, x^{**} \rangle = 0 \quad \forall x \in M \}.
\]
$X^{**}$ is the space of bounded, linear functionals from $X^*$ to $R$.

Note: In some cases, $X^{**} = X$, in this case, $X$ is a reflexive space.

Examples: (a) All Hilbert spaces are reflexive.

(b) $l_p$ spaces, $1 < p < \infty$ are reflexive.

Since $l_p^* = l_q$ and $l_p = l_q^* = l_p$.

But $l_p \neq l_1 + l_1^{**} = l_{\infty} \neq l_1$, hence only $1 < p < \infty$ are reflexive.

\[ \square \]

Remark: (i) if $\langle x, x^* \rangle = 0$, then $x$ is said to be orthogonal to $x^*$ even though they are in different spaces.

(ii) if $\langle x, x^* \rangle = \| x \| \| x^* \|$, then $x + x^*$ are said to be aligned.
Definition: Let \( M \subseteq X \), then
\[
\overset{\perp}{M} = \{ x \in X : \langle x, x' \rangle = 0 \}.
\]

Note the difference between \( \overset{\perp}{M} \) and \( M' \) when \( M \subseteq X \).

Thereon: let \( M \) be a subspace of a normed space \( X \). Then
\[
\overset{\perp}{(M')} = M.
\]

Proof:
\[
M' = \{ x' \in X' : \langle x, x' \rangle = 0 \forall x \in M \}.
\]
\[
\overset{\perp}{(M')} = \{ x \in X : \langle x, x' \rangle = 0 \forall x' \in M' \}.
\]
\[
\Rightarrow M \subseteq \overset{\perp}{(M')}.
\]

We have to prove that \( x \notin M \) does not belong to \( \overset{\perp}{(M')} \).

Let \( x \notin M \). Define \( f \) on \( [x+M] \) as
\[
f(m+dx) = \alpha
\]
Clearly \( f(M) = 0 \Rightarrow \| f \| = \sup_{x \in M} \frac{|f|}{\|f\|} = \| f_x \| \| \alpha \| \leq \infty \) (since \( M \) is closed).

Extend \( f \) to \( X \), call it \( x' \).
\[
\| x' \| = \| f \|_{[x+M]}
\]
\[ \Rightarrow x^* \in X^* \quad \text{and} \quad x^* \in M^\perp \quad \text{since} \]

\[ x^*(m) = x^*(m + 0 \cdot x) = 0 \]

But \[ x^*(x) = x^*(m + 1 \cdot x) = f(m + 1 \cdot x) = 1 \neq 0 \]

\[ \langle x, x^* \rangle = 0 \Rightarrow x \notin M^\perp. \]

The "extension" of the projection theorem to Banach spaces

**Theorem:** Let \( M \subseteq X \) be a subspace and let \( x \notin M \). Then, \( x \) solves

\[ \inf_{m \in M} \| x - m \| = \circ \]

iff there exists \( 0 \neq x^* \in M^\perp \) s.t.

\[ x - m_0 \text{ is aligned with } x^*. \]

(Note: the theorem does not say anything about existence or uniqueness.)

**Proof:**

(\( \Rightarrow \)) Suppose \( m_0 \) solves \( \circ \)

let \( x^* \in M^\perp \). Then

\[ \langle x, x^* \rangle = \langle x - m_0, x^* \rangle \]
\[ \leq \| x - m_0 \| \| x^* \| \]

from the definition of \( \| x^* \| \).

We have to demonstrate an \( x^* \) s.t.
\[ \langle x, x^* \rangle = \| x - m_0 \| \| x^* \| \]

Let
\[ f(x + \alpha m) = \alpha \| x - m_0 \|, \quad \forall \alpha \in \mathbb{R} \]

Thus, \( f \) is defined over \([x + M]\).

\[ \| f \|_{[x + M]} = \sup_{\alpha, m} \frac{\alpha \| x - m_0 \|}{\| m + \alpha x \|} \]

\[ = \sup_{\alpha, m} \frac{\| x - m_0 \|}{\| m + \alpha x \|} \]

\[ = \\| x - m_0 \| \inf_{m \in M} \frac{1}{\| m + x \|} \]

\[ = 1 \]

Extend \( f \) to \( X \), call it \( x^* \) s.t.
\[ \| x^* \| = 1 \]

\[ \langle x, x^* \rangle = f(x) = \| x - m_0 \| \]

\[ = \| x - m_0 \| \| x^* \|. \]

Since \( \| x^* \| = 1 \).
\( \iff \) Suppose \( \exists x \in M \) s.t.
\[
\langle x - m_0, x^* \rangle = \| x - m_0 \| \| x^* \|
\]

Note that \( \forall m \in M \)
\[
\langle x, x^* \rangle = \langle x - m, x^* \rangle
\]
\[
\leq \| x - m \| \| x^* \|
\]

But
\[
\langle x, x^* \rangle = \langle x - m_0, x^* \rangle
\]
\[
= \| x - m_0 \| \| x^* \|
\]
\[
\Rightarrow \| x - m_0 \| \leq \| x - m \| \quad \forall m \in M
\]

\[\Box\]

**Duality:** Sometimes minimization problem in \( x \) can be written as maximization problems in \( x^* \).

**Theorem:** Let \( M \) be a subspace of a normed space \( X \), and let \( x \notin M \).

\[
\inf_{m \in M} \| x - m \| = \max_{x^* \in M^*, \| x^* \| = 1} \langle x, x^* \rangle,
\]

and \( x - m_0 \) is aligned with \( x^* \). \( m_0 \) is the LHS.

**Proof:** Let \( d = \inf_{m \in M} \| x - m \| \)
\[
\Rightarrow \text{given } \epsilon > 0, \exists m \in M \text{ s.t.}
\]
\[ \| x - m_e \| \leq d + \varepsilon \]

If \( x^* \in M^\perp \) and \( \| x^* \| \leq 1 \), then

\[ \langle x, x^* \rangle = \langle x - m_e, x^* \rangle \]

\[ \leq \| x - m_e \| \| x^* \| \]

\[ \leq (d + \varepsilon) \]

\[ \Rightarrow \langle x, x^* \rangle \leq d \]

We have to demonstrate an \( x^* \) which achieves equality. This can be done as in the proof of the first part of the previous theorem. The rest of the proof is similar to the previous theorem as well. \( \square \)

**Existence:**

**Theorem:** Let subspace \( M \subset X \) (normed)

\[ \min \| x^* - m^* \| = \sup_{x \in M} \langle x, x^* \rangle \]

\[ m^* \in M^\perp \quad x \in M \]

\[ \| x \| \leq 1 \]

(In other words, the min on the LHS is achieved. Thus, min distance problem will have a solution if the space over which they are posed is the dual space of some other space.)

Further, if \( x_o \) achieves the sup on the RHS, then \( x^* - m^o \) is aligned with \( x_o \), where \( m^o \) is the solution to the LHS problem.
Proof: Similar to earlier proofs.

Geometric interpretation:

[x - m₀] is aligned with this element of \( M^⊥ \)

In a Hilbert space, note that

\[ \| x - m₀ \| = \text{the length of the projection of } x \text{ on to the unit vector in the direction } y. \]

But \( (x | y) = 0 < \frac{(x | y)}{\| y \|} \),

i.e., the projection in the \( y \) direction is maximum compared to projections on to other direction in \( M^⊥ \), establishing duality.