

## V. CONCLUSION

We proposed an explicit construction of fixed-length codes for SW source networks. The proposed code is linear and has two-step encoding and decoding procedures similar to a concatenated code used for channel coding. Further, if the sources are memoryless, the proposed code is universal and the probability of error vanishes exponentially as the block length tends to infinity. Regarding future research, we have the problem to obtain tight upper and lower bounds on the error exponent obtainable by the proposed code for DMSs.

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## On the Cost of Worst Case Coding Length Constraints

Dror Baron, *Student Member, IEEE*, and  
Andrew C. Singer, *Member, IEEE*

**Abstract**—We investigate the redundancy that arises from adding a worst case length constraint to uniquely decodable fixed-to-variable codes over achievable Huffman codes. This is in contrast to the traditional metric of the redundancy over the entropy. We show that the cost for adding constraints on the worst case coding length is small, and that the resulting bound is related to the Fibonacci numbers.

**Index Terms**—Data compression, Fibonacci numbers, Huffman coding, redundancy, source coding, uniquely decodable.

## I. INTRODUCTION

A fundamental tradeoff in lossless source coding is that some inputs can be compressed only if others are expanded. A reasonable objective is to compress well on average, while expanding little in the worst case.

The tradeoff between the expected coding length and the worst case coding expansion has received research attention. In [1], an algorithm for finding a code meeting these constraints is proposed, and in [2], the redundancy of the expected coding length over the entropy is bounded. In this correspondence, we investigate the redundancy of the expected coding length of constrained codes over that of achievable Huffman codes [3]. We bound this redundancy by a term that decays exponentially in the worst case coding expansion, and note that this term is related to the Fibonacci numbers. The problem is stated in Section II, the main results are given in Section III, and a discussion is provided in Section IV.

## II. PROBLEM FORMULATION

Consider a discrete alphabet  $\mathcal{X}$  and length- $N$  input sequences  $x$ , i.e.,  $x \in \mathcal{X}^N$ . We define a *source code*  $C$  as a mapping  $C: \mathcal{X}^N \rightarrow \mathcal{X}^*$  where  $\mathcal{X}^*$  is the set of finite-length sequences over  $\mathcal{X}$ . Following [4], let  $C(x)$  be the *codeword* corresponding to  $x$ , and  $l(x)$  the *length* of  $C(x)$ . The *expected coding length*  $L(C)$  for a random variable  $X$  with a probability mass function (PMF)  $p(x)$  is defined as

$$L(C) \triangleq \sum_{x \in \mathcal{X}^N} p(x)l(x). \quad (1)$$

A *uniquely decodable* code is a source code  $C$  with a dual mapping  $\tilde{C}: \mathcal{X}^* \rightarrow \mathcal{X}^N$ , such that  $\tilde{C}(C(x)) = x, \forall x \in \mathcal{X}^N$ . We say that the code  $C$  *expands*  $x$  when  $l(x) > N$ , and we define the *worst case coding expansion* as

$$W(C) \triangleq \max_{x \in \mathcal{X}^N} \{l(x) - N\} \quad (2)$$

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The authors are with the Coordinated Science Laboratory and the Department of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801 USA (e-mail: dbaron@uiuc.edu; acsinger@uiuc.edu).

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(it is nonnegative because no uniquely decodable code compresses all input sequences). Given the PMF  $p(x)$ ,  $x \in \mathcal{X}^N$ , and an (integer) constraint  $\Delta$  on the worst case coding expansion, the *constrained expected coding length* is defined as

$$L_\Delta^* \triangleq \min_{C: W(C) \leq \Delta} \{L(C)\}. \quad (3)$$

When the constraint is relaxed, and  $\Delta$  is increased, the constrained expected coding length will go down until at some stage it is equal to the unconstrained expected coding length. This is the expected coding length of the Huffman code, and is denoted by  $L_\infty^*$ . In the following section we bound  $L_\Delta^* - L_\infty^*$ .

### III. RESULTS

Any Huffman code can be viewed as a tree, where codewords correspond to *leaves*, and their prefixes correspond to *internal nodes*. The *nodes*, be they leaves or internal nodes, make up a *Huffman tree*. A leaf corresponds to some input  $x$  and its probability  $p(x)$ ; an internal node corresponds to a set of descendant leaves, and the total probability of those leaves. Lemma 1 bounds the probabilities corresponding to internal nodes, and is related to [5, Theorem 7], which bounds the probabilities corresponding to leaves. Both proofs are by induction, the only difference is due to initial conditions.

*Lemma 1:* Any depth- $k$  internal node in a Huffman tree corresponds to a set of codewords with total probability  $p_k$  satisfying

$$p_k \leq \frac{1}{f_k} \quad (4)$$

where  $f_{n+1} \triangleq f_n + (|\mathcal{X}| - 1)f_{n-1}$ , with initial conditions  $f_0 \triangleq 1$ ,  $f_1 \triangleq 2 - \frac{1}{|\mathcal{X}|}$ .

*Proof:* In the Huffman tree, let  $p_k, p_{k-1}, \dots, p_0 = 1$  be the probabilities corresponding to internal nodes on the path from our depth- $k$  node, denoted by  $\alpha$ , to the root. Let  $q_i^i, i \in \{1, \dots, |\mathcal{X}| - 1\}$  be the corresponding probability of node  $i$  merged with  $p_l$  into  $p_{l-1}$ . We prove  $p_l \geq f_{k-l} p_k$  by induction on  $l$ . First,  $p_k = 1 \cdot p_k = f_0 p_k$ . Second, the lemma requires  $\alpha$  to be an internal node, so at least one of its descendant nodes corresponds to at least a probability of  $\frac{1}{|\mathcal{X}|} p_k$ . But  $\alpha$  and its parent are internal nodes, so when the parent is created in the Huffman algorithm, its descendants are nodes corresponding to the minimal probabilities among all the nodes at that stage, so  $q_k^i \geq \frac{1}{|\mathcal{X}|} p_k$ . Therefore,

$$p_{k-1} = p_k + \sum_{i=1}^{|\mathcal{X}|-1} q_k^i \quad (5)$$

$$\begin{aligned} &\geq p_k + (|\mathcal{X}| - 1) \left( \frac{p_k}{|\mathcal{X}|} \right) \\ &= f_1 p_k \end{aligned} \quad (6)$$

where (5) is the merging of corresponding probabilities. The inductive step is

$$p_{l-1} = p_l + \sum_{i=1}^{|\mathcal{X}|-1} q_l^i \quad (7)$$

$$\geq p_l + (|\mathcal{X}| - 1) p_{l+1} \quad (8)$$

$$\geq (f_{k-l} + (|\mathcal{X}| - 1) f_{k-l-1}) p_k \quad (9)$$

$$= f_{k-l+1} p_k \quad (10)$$

where (7) is similar to (5), the reasoning in the second step leads to (8), (9) is by induction, and (10) uses the definition of  $f$ . The lemma (4) follows because  $p_0 = 1$ .  $\square$

For  $|\mathcal{X}| = 2$ ,  $f_n$  are essentially scaled Fibonacci numbers, i.e.,  $f_0 = \frac{2}{2}$ ,  $f_1 = \frac{3}{2}$ ,  $f_2 = \frac{5}{2}$ ,  $f_3 = \frac{8}{2}$ , etc. In the general case, the recursion for  $f_n$  can be solved using methods for difference equations [6] leading to

$$\begin{aligned} f_n &= \left( \frac{1}{2} + \frac{\frac{3}{2} - \frac{1}{|\mathcal{X}|}}{\sqrt{4|\mathcal{X}| - 3}} \right) \left( \frac{1 + \sqrt{4|\mathcal{X}| - 3}}{2} \right)^n \\ &\quad + \left( \frac{1}{2} - \frac{\frac{3}{2} - \frac{1}{|\mathcal{X}|}}{\sqrt{4|\mathcal{X}| - 3}} \right) \left( \frac{1 - \sqrt{4|\mathcal{X}| - 3}}{2} \right)^n, \quad n \geq 0. \end{aligned} \quad (11)$$

*Theorem 1:* Given a random variable  $X$  with a PMF  $p(x)$  over  $\mathcal{X}^N$ , then for  $\Delta > 0$

$$L_\Delta^* - L_\infty^* \leq \frac{1}{f_{\Delta-1}}. \quad (12)$$

*Proof:* We begin with  $C_H$ , a Huffman code that achieves  $L_\infty^*$ . We create a new code  $\tilde{C}$  (not necessarily a Huffman code) with length function  $\tilde{l}(x)$  and expected coding length  $L(\tilde{C})$ , by modifying  $C_H$  in two steps.

**Step 1:** Prune  $X_1 = \{x: l(x) \geq N + \Delta\}$  from the Huffman tree of  $C_H$ . If  $X_1$  is empty we are done, else nodes were pruned off a depth- $N + \Delta - 1$  internal node, so there exists a depth- $\Delta - 1$  internal node, including the one in the path from the depth- $N + \Delta - 1$  internal node to the root.

**Step 2:** Take any depth- $\Delta - 1$  internal node in the tree; denote it by  $\beta$ . Let  $X_2$  be the descendant leaves of  $\beta$ . Replace  $\beta$  with a new node  $\gamma$  with two descendants. The first descendant is  $\beta$ , and the second descendant is a depth- $N$  full tree with up to  $|\mathcal{X}|^N$  leaves that can accommodate all of  $X_1$ , since  $|X_1| < |\mathcal{X}|^N$ . The full tree for  $X_1$  starts at depth  $\Delta$  and goes up to depth  $N + \Delta$ , so  $\tilde{l}(x) \leq N + \Delta, \forall x \in X_1$ . For  $X_2$ ,  $l(x) < N + \Delta, \forall x \in X_2$ , so adding an additional symbol gives  $\tilde{l}(x) \leq N + \Delta, \forall x \in X_2$ . Therefore,  $W(\tilde{C}) \leq \Delta$ . The structure  $\beta$  originally resided at depth  $\Delta - 1$  in the tree, so by Lemma 1

$$\sum_{x \in X_2} p(x) \leq \frac{1}{f_{\Delta-1}}.$$

Therefore,  $\tilde{C}$  satisfies

$$\begin{aligned} L(\tilde{C}) &= \sum_{x \in \mathcal{X}^N} p(x) \tilde{l}(x) \\ &\leq \sum_{x \in X_1} p(x) \tilde{l}(x) + \sum_{x \in X_2} p(x) \tilde{l}(x) + \sum_{x \notin X_1 \cup X_2} p(x) \tilde{l}(x) \end{aligned} \quad (13)$$

$$\begin{aligned} &\leq \sum_{x \in X_1} p(x) l(x) + \sum_{x \in X_2} p(x) (l(x) + 1) + \sum_{x \notin X_1 \cup X_2} p(x) l(x) \end{aligned} \quad (14)$$

$$\begin{aligned} &= \sum_x p(x) l(x) + \sum_{x \in X_2} p(x) \\ &\leq L(C_H) + \frac{1}{f_{\Delta-1}} \end{aligned} \quad (15)$$

where (13) is an equality only if  $X_1$  and  $X_2$  are disjoint, and (14) arises because codewords in  $X_1$  became shorter. The result is obtained by noting that  $L(C_H) = L_\infty^*$ .  $\square$

### IV. DISCUSSION

We begin with several technical remarks on Theorem 1. First, the theorem does not apply for  $\Delta = 0$ , because there is no depth- $\Delta - 1$  internal node that can be split. In fact, for  $\Delta = 0$  there is no expansion, nor is there any compression, thus,  $L_0^* = N$ . Second, the theorem upper-bounds  $L_\Delta^* - L_\infty^*$ , but we cannot give a lower bound, because for a uniform PMF we have  $L_\Delta^* = N$ , hence  $L_\Delta^* - L_\infty^* = 0$ . Third, we can get a stronger bound on the expected coding length for  $\Delta = 2$ . In

this case, there always exists some depth- $\Delta - 1$  node (not necessarily internal) with

$$\sum_{x \in \mathcal{X}_2} p(x) \leq \frac{1}{|\mathcal{X}|}$$

so

$$L_2^* - L_\infty^* \leq \frac{1}{|\mathcal{X}|}. \quad (16)$$

Although Lemma 1 bounds the probabilities corresponding to depth- $k$  internal nodes, there could be nodes at that depth that correspond to even smaller probabilities.

The constructive method used in the proof of the theorem can be used to derive codes that satisfy constraints on the worst case coding expansion, but these are not necessarily optimal codes. However, the theorem is useful because it bounds the cost of the constraint by a term that decays exponentially in the expansion. A tighter bound in the main theorem could be obtained by finding a stronger version of Lemma 1 for the depth- $k$  node that corresponds to the smallest probabilities.

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## The Effect of Redundancy on Measurement

Oliver M. Collins, *Senior Member, IEEE*, and  
N. Vasudev, *Student Member, IEEE*

**Abstract**—This correspondence demonstrates that increasing the number of measurements made on a system beyond the minimum (i.e., the number of degrees of freedom in the system) can reduce the effect of measurement errors. The correspondence shows, for three broad classes of measurement problems, that, as the measurement redundancy increases, the residual error falls to a small constant value. Twice the number of degrees of freedom will allow performance very close to the ultimate limits. The correspondence presents one example within each class; however, a vast number of other measurement problems have a formulation identical to one of these three. The first example is the calibration of a resistive voltage divider consisting of a number of nominally equal resistors in series. The second is the determination of the complex response of a filter whose output can be observed only through a power detector. The third is the calibration of a resistive current combiner or current mode digital-to-analog converter (DAC). The basic ideas in the correspondence are general and extensions to other measurement problems outside of the three categories should be straightforward, although the details of the solution will be problem dependent.

**Index Terms**—Calibration, differential measurement, filter response, linearity standard, redundancy, voltage divider.

#### I. INTRODUCTION AND SUMMARY

The parameters of a mechanical or electrical system are usually determined by making as many measurements as there are degrees of freedom. The required system parameter accuracy then determines the measurement precision. This correspondence shows that making more than the minimum number of measurements may be a much more efficient technique than improving the precision of the individual measurements. The possible systems to which this technique may be applied are too diverse to be captured by a simple and compact analytical formulation. Thus, this correspondence develops techniques for three different measurement models, scalar difference measurements, complex measurements, and scalar additive measurements. Each has both compact mathematical formalization and wide practical applicability, and many different practical problems will fit within each model's framework. The correspondence presents one detailed practical example for each of the three models.

The first practical example is that of generating an absolute standard of linearity or absolute ratio standard using a resistive voltage divider [1]–[4]. The physical setup for the measurement is shown in Fig. 1 and uses a nanovoltmeter and a stable variable voltage source. The voltage source and the nanovoltmeter are connected in series. This combination is hooked up to the two points across which the potential difference is being measured. The variable voltage source is adjusted to obtain a measurement in the linear range of the nanovoltmeter. The voltage source is not calibrated, so the only measurements that can be made are the differences between two nearly equal voltages. The resistors are all nominally identical; however, each has a small error. Thus, the voltages on the terminals are not precise. The purpose of the calibration

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O. M. Collins is with the Electrical Engineering Department, University of Notre Dame, Notre Dame, IN 46556 USA (e-mail: Oliver.M.Collins.62@nd.edu).

N. Vasudev was with the Electrical Engineering Department, University of Notre Dame, Notre Dame, IN 46556 USA. He is now with Stryker Endoscopy Corporation, Santa Clara, CA 95051 USA.

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