Linear Equalization Via Factor Graphs

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Abstract — **We apply the factor graph framework to the techniques of linear equalization and decision feedback equalization to obtain a new class of low complexity equalization algorithms. The estimation of Gaussian processes has been studied in previous work, and the application of factor graphs to this problem is a recent extension. Here we use a factor graph model for the specific estimation problem of equalization and use the sum-product algorithm to obtain the desired estimate. We derive reduced complexity message passing update equations and detail the complexity of the resulting algorithms.**

I. Summary

We consider the use of factor graphs [1] to estimate a transmitted signal over a linear, possibly time-varying, channel with additive white noise. Let $x = (x_1, x_2, \ldots, x_L) \in \mathcal{B}^L$ be a transmitted signal consisting of independent symbols from some alphabet $\mathcal{B}$ with a priori probability density functions $f(x_k)$. Let $\mathbf{x}_k = [x_k \; x_{k-1} \ldots x_{k-N+1}]^T$ and $\mathbf{h}_k \in \mathbf{R}^N$, $1 \le k \le L+N-1$, where $x_k = 0$ for $k \le 0$ or $k \ge L+1$. Then we define the observation $y_k = \mathbf{h}_k^T \mathbf{x}_k + w_k, 1 \le k \le L+N-1$, where w_k is zero mean white noise independent of x and with variance σ_k^2. The goal then is to form an estimate $\hat{x}$ of x from the received signal $y = (y_1, y_2, \ldots, y_{L+N-1})$.

Directly applying the factor graph framework to obtain the maximum a posteriori (MAP) estimate of x would result in an algorithm with complexity that is exponential in N. To obtain reduced complexity algorithms, we consider instead factor graphs that model approximate a posteriori distributions that are jointly Gaussian with the same first and second order statistics as the true distributions of interest. We use the sum-product algorithm to obtain the MAP estimate of x from this approximate (Gaussian) distribution [2]. This estimate is also the linear minimum mean square error estimate of x based on the approximate distribution as well as based on the true distribution.

The first equalizer we examine is an unconstrained linear equalizer (LE). We consider a factorization of the Gaussian approximation $\hat{f}(x|y)$ of the true a posteriori distribution $f(x|y)$, as depicted in Figure 1, where $\hat{f}(x_k)$ is a Gaussian distribution with the same mean and variance as $f(x_k)$ and $\hat{f}(y_k|\mathbf{x}_k)$ is a conditional Gaussian distribution with conditional mean $\mathbf{h}_k^T\mathbf{x}_k$ and variance σ_k^2. We obtain the estimate $\hat{x}_k$ as the mean of the marginal distribution $\hat{f}(x_k|y)$. By deriving the message-passing updates, we show that the resulting algorithm has complexity that is $\mathcal{O}(LN^3)$. Use of two well-known matrix inversion formulas reduces this to $\mathcal{O}(LN^2)$. These formulas can be used to reduce the complexity of the remaining algorithms to be presented as well.

We next consider a constrained LE. The factor graph algorithm is similar to the unconstrained case, but instead is implemented on the factor graph corresponding to the Gaussian approximation $\hat{f}(x|\mathbf{y}_k)$ of the true a posteriori distribution $f(x|\mathbf{y}_k)$, where $\mathbf{y}_k$ is a vector containing a subset of the elements of y. In essence, the algorithm operates on a truncated factor graph. So, each estimate $\hat{x}_k$ is found by computing the marginal distribution $f(x_k|\mathbf{y}_k)$ on the corresponding truncated factor graph.

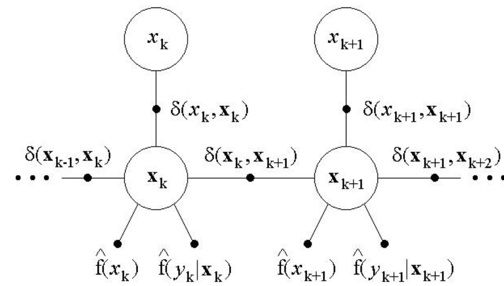

Fig. 1: Factor graph model for unconstrained linear equalization.

In addition to describing the constrained LE algorithm, we make several observations. First, this equalizer can be interpreted as an implementation of the unconstrained LE using a particular, sub-optimal (in the sense of the unconstrained LE) message-passing schedule. Second, recursive updates exist that allow for more efficient calculation of $\hat{x}_{k+1}$ given the messages that were computed when calculating $\hat{x}_k$. Lastly, with a minor modification to the algorithm, the filter coefficients of the LE can be explicitly obtained in addition to the data estimate $\hat{x}$. This is particularly useful when the channel is time-invariant and the a priori distribution $f(x_k)$ is the same for all k.

Finally, we consider implementing a decision feedback equalizer on a factor graph. In this framework, the process of decision feedback is interpreted as an "on-the-fly" modification to the factor graph that is used for constrained linear equalization to reflect the assumed absolute certainty in the decided bits. We present the resulting algorithm, describing subtle issues that arise in this framework.

The presentation of these algorithms is meant to broaden the class of problems that factor graphs can be used to solve. That factor graphs can be used to derive these much studied equalization algorithms suggests the possibility that other equalization techniques not previously considered might be discovered in the factor graph framework. This is an avenue of future research.

References

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