Derivation of Mean and Covariance Update Using SKL

Ruei-sung Lin David Ross Jongwoo Lim Ming-Hsuan Yang

Oct 21, 2003

Let $X = \{x_1, x_2, \ldots, x_N\}$, $Y = \{x_{N+1}, x_{N+2}, \ldots, x_{N+M}\}$ and $Z = [X|Y]$. Denote the mean and the covariance of X, Y, and Z as:

$$
\begin{aligned}
m_x &= \frac{1}{N}\sum_{i=1}^{N} x_i \\
m_y &= \frac{1}{M}\sum_{i=N+1}^{N+M} x_i \\
m_z &= \frac{1}{N+M}\sum_{i=1}^{N+M} x_i \\
C_{xx} &= \frac{1}{N}\sum_{i=1}^{N}(x_i - m_x)(x_i - m_x)^T \\
C_{yy} &= \frac{1}{M}\sum_{i=1}^{k}(x_i - m_y)(x_i - m_y)^T \\
C_{zz} &= \frac{1}{N+M}\sum_{i=1}^{N+k}(x_i - m_z)(x_i - m_z)^T
\end{aligned}
$$

Based on the definition above, we know:

$$
m_z = \frac{N}{N+M}m_x + \frac{M}{N+M}m_y
$$

Therefore,

$$
\begin{aligned}
m_x - m_z &= \frac{M}{N+M}(m_x - m_y) \\
m_y - m_z &= \frac{N}{N+M}(m_y - m_x)
\end{aligned}
$$

Also, it can be shown that:

$$C_{zz} = \frac{N}{N+M} C_{xx} + \frac{M}{N+M} C_{yy} + \frac{NM}{(N+M)^2} (m_y - m_x)(m_y - m_x)^T$$

Proof:

$$\begin{aligned}
(N+M)C_{zz} &= \sum_{i=1}^{N} (x_i - m_x + m_x - m_z)(x_i - m_x + m_x - m_z)^T + \sum_{i=N+1}^{N+k} (x_i - m_z)(x_i - m_z)^T \\
&= NC_{xx} + N(m_x - m_z)(m_x - m_z)^T + \sum_{i=N+1}^{N+k} (x_i - m_y + m_y - m_z)(x_i - m_y + m_y - m_z)^T \\
&= NC_{xx} + N(m_x - m_z)(m_x - m_z)^T + MC_{yy} + M(m_y - m_z)(m_y - m_z)^T \\
&= NC_{xx} + MC_{yy} + \frac{NM^2}{(N+M)^2} (m_x - m_y)(m_x - m_y)^T + \frac{N^2 M}{(N+M)^2} (m_x - m_y)(m_x - m_y)^T \\
&= NC_{xx} + MC_{yy} + \frac{NM}{N+M} (m_x - m_y)(m_x - m_y)^T
\end{aligned}$$

Therefore,

$$C_{zz} = \left[\sqrt{\tfrac{N}{N+M}} X \middle| \sqrt{\tfrac{M}{N+M}} Y \middle| \sqrt{\tfrac{NM}{(N+M)^2}} (m_x - m_y) \right] \left[\sqrt{\tfrac{N}{N+M}} X \middle| \sqrt{\tfrac{M}{N+M}} Y \middle| \sqrt{\tfrac{NM}{(N+M)^2}} (m_x - m_y) \right]^T$$

Let $\gamma = \frac{N}{N+M}$. Then, $\frac{M}{N+M} = 1 - \frac{N}{N+M} = (1-\gamma)$

$$C_{zz} = \left[\sqrt{\gamma} X \middle| \sqrt{(1-\gamma)} Y \middle| \sqrt{\gamma(1-\gamma)} (m_x - m_y) \right] \left[\sqrt{\gamma} X \middle| \sqrt{(1-\gamma)} Y \middle| \sqrt{\gamma(1-\gamma)} (m_x - m_y) \right]^T$$