Opportunistic Orthogonal Writing on Dirty Paper

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The Dirty Paper Channel

\[ y(t) = x(t) + s(t) + w(t) \]

WGN

p.s.d. \( \frac{N_0}{2} \)
The Dirty Paper Channel

\[ y(t) = x(t) + s(t) + w(t) \]

WG Interference \[ \text{p.s.d. } \frac{N_s}{2} \]

WGN \[ \text{p.s.d. } \frac{N_0}{2} \]
The Dirty Paper Channel

\[ y(t) = x(t) + s(t) + w(t) \]

- **Power** $P$
- **Bandwidth** $W$
- **WG Interference**
- **WGN**

\[ \text{p.s.d. } \frac{N_s}{2} \]
\[ \text{p.s.d. } \frac{N_0}{2} \]

- Observation (Cos81): Capacity of the channel is

\[ W \log \left( 1 + \frac{P}{N_0 W} \right) \]

- An abstract binning scheme
Focus of this talk

\[ y(t) = x(t) + s(t) + w(t) \]

- Power \( P \)
- Wideband dirty paper channel
- WG Interference
- WGN
- \( s(t) \) p.s.d. \( \frac{N_s}{2} \)
- \( w(t) \) p.s.d. \( \frac{N_0}{2} \)
Main Result

- Setting: Wideband channel
  - Criterion: Minimum Energy per reliable Bit $\mathcal{E}_b$

- Main Result:
  - An explicit binning scheme
  - Ramifications:
    * Typical error events
    * Low rate VQ for wideband Gaussian source
    * Generalization to abstract alphabets
PPM and Wideband AWGN Channel

$M$ Pulses

- Each message corresponds to a pulse
Opportunistic PPM

$K$ Sub-pulses

$M$ Pulses

◊ Each message corresponds to $K$ sub-pulses
Encoding Opportunistic PPM

- Transmit sub-pulse (among $K$ possible choices) where $s(t)$ is largest
Decoding Opportunistic PPM

\[ y(t) \]

◊ Choose sub-pulse (among \( MK \) possible choices) where \( y(t) \) is largest
Relieving the Tension

- Transmit signal amplitude

\[ \sqrt{\mathcal{E}_s} = \sqrt{\mathcal{E}_b \log M} + \sqrt{N_s \log K} \]

- \( \sqrt{N_s \log K} \) is the opportunistic amplitude gain

- Larger \( K \) means larger signal amplitude

- But decoder has more choices to get confused (\( MK \) sub-pulses)
Relieving the Tension

◊ Tranmit signal amplitude
\[ \sqrt{\mathcal{E}_s} = \sqrt{\mathcal{E}_b \log M} + \sqrt{N_s \log K} \]

◊ Error probability
\[ \mathbb{P}[\text{Error}] \leq MK \quad \mathbb{P}\left[s + w \geq \sqrt{\mathcal{E}_s}\right] \]
Relieving the Tension

◊ Transmit signal amplitude

\[ \sqrt{\mathcal{E}_s} = \sqrt{\mathcal{E}_b \log M} + \sqrt{N_s \log K} \]

◊ Error probability

\[ \mathbb{P} \left[ \text{Error} \right] \leq MK \mathbb{P} \left[ s + w \geq \sqrt{\mathcal{E}_s} \right] \]

\[ \approx MK \mathbb{P} \left[ s \geq \frac{N_s}{N_s + N_0} \sqrt{\mathcal{E}_s} \right] \mathbb{P} \left[ w \geq \frac{N_0}{N_s + N_0} \sqrt{\mathcal{E}_s} \right] \]
Relieving the Tension

◊ Transmit signal amplitude

\[ \sqrt{\mathcal{E}_s} = \sqrt{\mathcal{E}_b \log M} + \sqrt{N_s \log K} \]

◊ Error probability

\[ \mathbb{P} \left[ \text{Error} \right] \leq MK \mathbb{P} \left[ s + w \geq \sqrt{\mathcal{E}_s} \right] \]

\[ \approx MK \mathbb{P} \left[ s \geq \frac{N_s}{N_s + N_0} \sqrt{\mathcal{E}_s} \right] \mathbb{P} \left[ w \geq \frac{N_0}{N_s + N_0} \sqrt{\mathcal{E}_s} \right] \]

\[ \approx M \mathbb{P} \left[ w \geq \sqrt{\mathcal{E}_b \log M} \right] \]

◊ If possible, then desired result follows
Choice of $K$

- Transmit signal amplitude

$$\sqrt{E_s} = \sqrt{E_b \log M} + \sqrt{N_s \log K}$$

- Choose $\log K = \frac{N_s}{N_0} \log M$

- Then

$$\frac{N_s}{N_s + N_0} \sqrt{E_s} = \sqrt{N_s \log K}, \quad \frac{N_0}{N_s + N_0} \sqrt{E_s} = \sqrt{E_b \log M}$$
Tension at correct choice of $K$

- Transmit signal amplitude $\sqrt{\mathcal{E}_s} = \sqrt{\mathcal{E}_b \log M} + \sqrt{N_s \log K}$

- Choose $\log K = \frac{N_s}{N_0} \log M$

\[
\frac{N_s}{N_s + N_0} \sqrt{\mathcal{E}_s} = \sqrt{N_s \log K}, \quad \frac{N_0}{N_s + N_0} \sqrt{\mathcal{E}_s} = \sqrt{\mathcal{E}_b \log M}
\]

- Error probability

\[
P[\text{Error}] \leq MK \mathbb{P} \left[ s \geq \frac{N_s}{N_s + N_0} \sqrt{\mathcal{E}_s} \right] \mathbb{P} \left[ w \geq \frac{N_0}{N_s + N_0} \sqrt{\mathcal{E}_s} \right]
\]

\[
= MK \mathbb{P} \left[ s \geq \sqrt{N_s \log K} \right] \mathbb{P} \left[ w \geq \sqrt{\mathcal{E}_b \log M} \right]
\]
The AWGN Performance

- Transmit signal amplitude: \( \sqrt{E_s} = \sqrt{E_b \log M} + \sqrt{N_s \log K} \)

- Choose \( \log K = \frac{N_s}{N_0} \log M \)

\[
\frac{N_s}{N_s + N_0} \sqrt{E_s} = \sqrt{N_s \log K}, \quad \frac{N_0}{N_s + N_0} \sqrt{E_s} = \sqrt{E_b \log M} 
\]

- Error probability

\[
P[\text{Error}] \leq MK \ P \left[ s \geq \frac{N_s}{N_s + N_0} \sqrt{E_s} \right] \ P \left[ w \geq \frac{N_0}{N_s + N_0} \sqrt{E_s} \right] 
\]

\[
= MK \ P \left[ s \geq \sqrt{N_s \log K} \right] \ P \left[ w \geq \sqrt{E_b \log M} \right] 
\]

\[
\approx MK \ P \left[ w \geq \sqrt{E_b \log M} \right] 
\]

- AWGN performance is achieved
Non-Gaussian Dirt

- Interference $s(t)$ is not AWG
- Make length $N$ of each pulse long enough:

$$\frac{\sum_{i=1}^{N} s_i}{\sqrt{N}} \sim \mathcal{N}$$

- Essentially convert dirt into Gaussian
- **Robustness**: OPPM works over a (restricted) arbitrarily varying dirt
Story So Far

- OPPM is a simple concrete scheme to achieve AWGN performance over wideband Dirty Paper channel
Pedagogical Utility

- OPPM is a simple concrete scheme to achieve AWGN performance over wideband Dirty Paper channel

Utility of OPPM

- OPPM is a simple concrete scheme to achieve AWGN performance over wideband Dirty Paper channel


- OPPM is also useful to prove new results
Typical Error Events

- Error occurs in two ways:
Typical Error Events

- Error occurs in two ways:
  - Opportunistic gain is not large enough

\[ \mathbb{P} \left[ \max_{k=1\ldots K} s_k < \sqrt{N_s \log K} \right] \]
Typical Error Events

◊ Error occurs in two ways:

◊ Opportunistic gain is not large enough

\[ \mathbb{P} \left[ \max_{k=1 \ldots K} s_k < \sqrt{N_s \log K} \right] \]

◊ Interference plus Noise is too large

\[ \mathbb{P} \left[ s + w > \sqrt{\mathcal{E}_s} \right] \]
**Typical Error Events**

- Error occurs in two ways:
  - Opportunistic gain is not large enough
    \[
    \mathbb{P}\left[ \max_{k=1...K} s_k < \sqrt{N_s \log K} \right] \approx \exp^{-\epsilon_1 \log K}
    \]
  - Interference plus Noise is too large
    \[
    \mathbb{P}\left[ s + w > \sqrt{\mathcal{E}_s} \right] \approx \exp^{-\epsilon_2 \log K}
    \]
Typical Error Events

◊ Error occurs in two ways:

◊ Opportunistic gain is not large enough

\[ \mathbb{P} \left[ \max_{k=1 \ldots K} s_k < \sqrt{N_s \log K} \right] \approx \exp^{-\exp^{\epsilon_1 \log K}} \]

◊ Interference plus Noise is too large

\[ \mathbb{P} \left[ s + w > \sqrt{\mathcal{E}_s} \right] \approx \exp^{-\epsilon_2 \log K} \]

◊ Typical error event same as in wideband AWGN channel

Error exponent of wideband dirty paper channel same as that of AWGN channel
OPPM is Structured Binning

- Interpretation of Costa’s binning scheme
  - Each bin is a good quantizer
  - All bins put together, good channel code
OPPM is Structured Binning

- Interpretation of Costa’s binning scheme
  - Each bin is a good quantizer
  - All bins put together, good channel code

- OPPM fits this theme
  - Overall it is PPM - a good channel code

- Where is the quantizer?
Pulse Position Quantization

Source

Reconstruction
Optimality of PPQ

Consider performance criterion:

$$\max \quad \frac{\text{Quantization Rate}}{\text{Reduction in Distortion}}$$
Optimality of PPQ

- Consider performance criterion:
  \[
  \max \frac{\text{Quantization Rate}}{\text{Reduction in Distortion}}
  \]

- An optimality property:
  PPQ is the best low rate VQ of a wideband Gaussian source
A Deeper Understanding

◊ Opportunistic PPM is

a combination of PPM (a good wideband channel code) and PPQ (a good wideband VQ)
A Deeper Understanding

◊ Opportunistic PPM is

   a combination of PPM (a good wideband channel code)
   and PPQ (a good wideband VQ)

◊ Generalization:

   – Abstract alphabet Gelfand-Pinsker channel
   – Abstract cost measure over input alphabet
   – Performance measure:

       Capacity per unit cost of an abstract Gelfand-Pinsker channel
Gelfand-Pinsker Channel

- Memoryless Channel
- Alphabets: $x \in \mathcal{X}$, $s \in S$, $y \in \mathcal{Y}$
- Cost: $b(x)$ for $x \in \mathcal{X}$
Capacity per Unit Cost

- Largest rate of reliable communication per unit cost:
  \[
  \max_{P_{X|U,S}} \frac{I(U;Y) - I(U;S)}{\mathbb{E}[b(X)]}.
  \]

- \(U\) is an auxiliary random variable

- Achieved by an abstract binning scheme
Capacity per Unit Cost

- Largest rate of reliable communication per unit cost:
  \[ \max_{P_{XU|S}} \frac{I(U; Y) - I(U; S)}{\mathbb{E}[b(X)]}. \]
  
- \( U \) is an auxiliary random variable

- Achieved by an abstract binning scheme

- Main result:
  
  Generalized Opportunistic PPM achieves the capacity per unit cost

- Need a zero cost input alphabet
Generalized PPM

N is length of each pulse

M Pulses

◊ Achieves capacity per unit cost of a classical channel

Each message corresponds to $K$ sub-pulses
Encoding

Transmit $x = f(s)$ over sub-pulse that has type V

Deterministic function $f : \mathcal{S} \rightarrow \mathcal{X}$
Encoding

◊ Transmit $x = f(s)$ over sub-pulse that has type $V$

◊ Deterministic function $f : S \rightarrow X$

◊ Need $K = \exp^{D(V||S)}$
Decoding

- **Binary Hypothesis Test** at any sub-pulse

\[ \mathcal{H}_0 : \mathbb{P}_Y | X = f(V), V \]

\[ \mathcal{H}_1 : \mathbb{P}_Y | X = 0, S \]
Decoding

- **Binary Hypothesis Test** at any sub-pulse

  \[ H_0 : \mathbb{P}_{Y|X=f(V),V} \]
  \[ H_1 : \mathbb{P}_{Y|X=0,S} \]

- **Fix** \( \mathbb{P}[H_0 \rightarrow H_1] = \epsilon. \)

- **Stein’s Lemma (almost)**

  \[
  \mathbb{P}[H_0 \rightarrow H_0] \stackrel{\text{def}}{=} \exp^{-N \mathbb{D}(\mathbb{P}_{Y|X=f(V),V} || \mathbb{P}_{Y|X=0,S})}
  \]
Decoding

- **Binary Hypothesis Test** at any sub-pulse

  \[ \mathcal{H}_0 : \mathbb{P}_{Y|X = f(V), V} \]
  \[ \mathcal{H}_1 : \mathbb{P}_{Y|X = 0, S} \]

- Fix \( \mathbb{P}[\mathcal{H}_0 \rightarrow \mathcal{H}_1] = \epsilon \).

- Stein’s Lemma (almost)

  \[ \mathbb{P}[\mathcal{H}_0 \rightarrow \mathcal{H}_0] = \exp^{-N \mathcal{D}(\mathbb{P}_{Y|X = f(V), V} || \mathbb{P}_{Y|X = 0, S})} \]

- Independent tests for each sub-pulse
  - *MK* tests
Rate per Unit Cost

Error Probability

\[
\mathbb{P} \left[ \text{Error} \right] \leq M \ K \ \exp^{-ND\left(\mathbb{P}_{Y|X=f(V), V} \ || \mathbb{P}_{Y|X=0, S}\right)} \\
= M \ \exp^N \ D(V \| S) \ \exp^{-N \ D\left(\mathbb{P}_{Y|X=f(V), V} \ || \mathbb{P}_{Y|X=0, S}\right)} \\
= M \ \exp^{-N \left( D\left(\mathbb{P}_{Y|X=f(V), V} \ || \mathbb{P}_{Y|X=0, S}\right) - D(V \| S) \right)}
\]
Rate per Unit Cost

◊ Error Probability

\[ P[\text{Error}] \leq M K \exp^{-ND(\mathbb{P}_{Y|X=f(V),V||\mathbb{P}_{Y|X=0,S}})} \]

\[ = M \exp^{-N(D(\mathbb{P}_{Y|X=f(V),V||\mathbb{P}_{Y|X=0,S}})-D(V||S))} \]

◊ Cost incurred

\[ N \mathbb{E}[b(f(V))] \]
**Rate per Unit Cost**

- **Error Probability**

\[
P[\text{Error}] \leq M K \exp^{-N D\left(\mathbb{P}_{Y|X=f(V),V}||\mathbb{P}_{Y|X=0,S}\right)}
= M \exp^{-N \left(D\left(\mathbb{P}_{Y|X=f(V),V}||\mathbb{P}_{Y|X=0,S}\right) - D(V||S)\right)}
\]

- **Cost incurred**

\[
N\mathbb{E}[b(f(V))]
\]

- **Conclusion: achievable rate per unit cost**

\[
\log M \frac{D\left(\mathbb{P}_{Y|X=f(V),V}||\mathbb{P}_{Y|X=0,S}\right)}{N\mathbb{E}[b(f(V))]} = \frac{D\left(\mathbb{P}_{Y|X=f(V),V}||\mathbb{P}_{Y|X=0,S}\right) - D(V||S)}{\mathbb{E}[b(f(V))]}.
\]
Main Result

◇ **Theorem:** The capacity per unit cost is

\[
\max_{P_V \ll P_S, \ f:S \to X} \frac{D \left( P_{Y \mid X=f(V),V} \mid \mid P_{Y \mid X=0,S} \right) - D \left( V \mid S \right)}{\mathbb{E} \left[ b(f(V)) \right]}
\]

◇ Converse follows by calculus on

\[
\max_{P_{XU \mid S}} \frac{I(U;Y) - I(U;S)}{\mathbb{E} \left[ b(X) \right]}
\]

◇ Key idea: identify

\( V \) (the preferred type) from

\( U \) (the auxiliary random variable)
Hindsight

◇ Return to Dirty Paper Channel:

\[ y[m] = x[m] + s[m] + w[m] \]

◇ Cost function \( b(x) = x^2 \)

◇ Choose \( X = x \) and \( V = \frac{N_s}{N_0} x + S \)

◇ Then

\[
\frac{D \left( \mathbb{P}_{Y|X=x,V} \middle| \mathbb{P}_{Y|X=0,S} \right) - D \left( V \middle| S \right)}{x^2} = \frac{1}{N_0}
\]
A Summary

- Capacity of Gelfand-Pinsker Channels achieved by binning scheme

- Binning scheme
  - Each bin a good quantizer
  - Overall good channel code

- Consider cost-efficient quantization and channel coding
  - S. Verdú, “Capacity per Unit Cost”

- Combination of cost-efficient quantizers and channel codes

  achieves capacity per unit cost of Gelfand-Pinsker channel
Another Ramification

- Consider low-rate VQ of wideband Gaussian source $x_1(t)$
- Decoder has access to correlated wideband Gaussian source $x_2(t)$
- A type of Wyner-Ziv lossy compression
  - A binning scheme suggests no “rate loss”
- Ideas developed here suggest an explicit scheme: OPPQ
OPPQ: Encoding

Source

Reconstruction
without side information
OPPQ: Reconstruction

Source

Reconstruction with side information