Extremal Inequalities for Quadratic Multiterminal Information Theory

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Multi-terminal Information Theoretic Problems

Information Transmission

$m_1 \rightarrow (\hat{m}_1, \hat{m}_2) \rightarrow (\hat{x}_1, \hat{x}_2)$

Multiple Access
Multi-terminal Information Theoretic Problems

Information Transmission

\[ m_1 \rightarrow (\hat{m}_1, \hat{m}_2) \]

\[ m_2 \rightarrow (\hat{m}_1, \hat{m}_2) \]

Multiple Access

\[ (m_1, m_2) \rightarrow \hat{m}_1 \]

\[ (m_1, m_2) \rightarrow \hat{m}_2 \]

Broadcast
Multi-terminal Information Theoretic Problems

Information Transmission

Multi-terminal Information Theoretic Problems

Information Representation

Broadcast
Multi-terminal Information Theoretic Problems

Information Transmission

Multiple Access

Broadcast

Information Representation

Distributed Source Coding

Multiple Description
Multi-terminal Information Theoretic Problems

Information Transmission

\[ m_1 \rightarrow (\hat{m}_1, \hat{m}_2) \]

\[ m_2 \rightarrow (\hat{m}_1, \hat{m}_2) \]

Multiple Access

\[ (m_1, m_2) \rightarrow \hat{m}_1 \]

\[ (m_1, m_2) \rightarrow \hat{m}_2 \]

Broadcast

Information Representation

\[ x_1 \rightarrow (\hat{x}_1, \hat{x}_2) \]

\[ x_2 \rightarrow (\hat{x}_1, \hat{x}_2) \]

Distributed Source Coding

\[ x \rightarrow \hat{x}_1 \]

\[ x \rightarrow \hat{x}_2 \]

Multiple Description
Multi-terminal Information Theoretic Problems

Information Transmission

Multiple Access

Broadcast

Information Representation

Distributed Source Coding

Multiple Description
Game Plan

◦ Quadratic Multiterminal Problems
Game Plan

- Quadratic Multiterminal Problems

- Second-Order Constraints
  - Source and noise statistics
  - Transmit and reconstruction error signals
Game Plan

◊ Quadratic Multiterminal Problems

◊ Second-Order Constraints
  – Source and noise statistics
  – transmit and reconstruction error signals

◊ Derive Outer Bounds
Game Plan

- Quadratic Multiterminal Problems
- **Second-Order** Constraints
  - Source and noise statistics
  - transmit and reconstruction error signals
- Derive **Outer Bounds**
  - Extremal Inequalities
Game Plan

- Quadratic Multiterminal Problems

- Second-Order Constraints
  - Source and noise statistics
  - Transmit and reconstruction error signals

- Derive Outer Bounds
  - Extremal Inequalities

- For Gaussian signals, outer bound is tight
  - Natural inner bound
A Canonical Network Data Compression Problem

- Example: Sensor networks

- Multiple sensors observe a common environment
  - observations are correlated

- Need to compress observations
  - sensors work distributedly
  - at low rates

- Reconstruction by a fusion center
  - joint reconstruction
  - subject to distortion criterion
An Abstraction

$y_1^n \rightarrow \text{Enc 1}$

$y_2^n \rightarrow \text{Enc 2}$

$R_1 \rightarrow \hat{y}_1^n$

$R_2 \rightarrow \hat{y}_2^n$
An Abstraction

Sources $y_1(m)$ and $y_2(m)$ memoryless

Correlated Gaussian statistics
An Abstraction

Sources $y_1(m)$ and $y_2(m)$ memoryless with Gaussian statistics

Quadratic distortion constraints

$$\sum_{m=1}^{n} E \left[ (y_i(m) - \hat{y}_i(m))^2 \right] \leq nd_i, \quad \text{for } i = 1, 2$$
An Abstraction

\[ y_1^n \rightarrow \text{Enc 1} \rightarrow R_1 \rightarrow \hat{y}_1^n \]
\[ y_2^n \rightarrow \text{Enc 2} \rightarrow R_2 \rightarrow \hat{y}_2^n \]

- **Sources** $y_1(m)$ and $y_2(m)$ memoryless with Gaussian statistics
- **Quadratic** distortion constraints
- **Question:** What are optimal distributed compression schemes?
Multiterminal Information Theory

- Classical open question in network information theory
  - Multiterminal distributed source coding
Multiterminal Information Theory

- Classical open question in network information theory
  - Multiterminal distributed source coding
- Performance criterion: asymptotic rates of compression
Multiterminal Information Theory

- Classical open question in network information theory
  - Multiterminal distributed source coding
- Performance criterion: asymptotic rates of compression
- Formal question:

  What is the rate-distortion region?
Multiterminal Information Theory

- Classical open question in network information theory
  - Multiterminal distributed source coding
- Performance criterion: asymptotic rates of compression
- Formal question:
  
  What is the rate-distortion region?

- Long-standing open problem
Lossless Distributed Compression

- Discrete sources of information
- Distortions tolerated: $d_1 = d_2 = 0$

- Classical result of Slepian and Wolf: (1973)
  
  a binning scheme ensures no-loss distributed compression
Lossless Distributed Compression

- Discrete sources of information
- Distortions tolerated: $d_1 = d_2 = 0$

- Classical result of Slepian and Wolf: (1973)
  
a binning scheme ensures no-loss distributed compression

- Suggests a natural strategy for lossy compression:
  - First, digitize the analog source into bits
  - Second, use Slepian-Wolf binning to convey the bits losslessly
A Natural Separation Scheme

\[ y_1^n \xrightarrow{\text{VQ–1}} \text{Binning Scheme–1} \xrightarrow{R_1} \hat{y}_1^n \]

\[ y_2^n \xrightarrow{\text{VQ–2}} \text{Binning Scheme–2} \xrightarrow{R_2} \hat{y}_2^n \]
A Natural Gaussian Separation Scheme

VQ-1 and VQ-2 are Gaussian quantizers
Optimality of Natural Separation Scheme

- Gaussian separation scheme evaluated by Berger and Tung (1978)
Optimality of Natural Separation Scheme

- Gaussian separation scheme evaluated by Berger and Tung (1978)

- Optimality unresolved
  - one of several longstanding open problems
Main Result

- The Gaussian separation scheme is optimal for Gaussian sources.
Main Result

- The Gaussian separation scheme is \textit{optimal} for Gaussian sources

- \textbf{Robustness:}

  - Gaussian statistics is the \textit{worst-case} scenario for this scheme.
    - among sources with same second order statistics
Main Result

- The Gaussian separation scheme is **optimal** for Gaussian sources.

- **Robustness:**
  - Gaussian statistics is the **worst-case** scenario for this scheme.
    - among sources with same second order statistics
  - Scheme optimal when terminals are **remote**
Remote Terminals

\[
\begin{aligned}
    y_1^n &\rightarrow + & z_1^n &\rightarrow Enc 1 & R_1 &\rightarrow \hat{y}_1^n \\
\end{aligned}
\]

\[
\begin{aligned}
    y_2^n &\rightarrow + & z_2^n &\rightarrow Enc 2 & R_2 &\rightarrow \hat{y}_2^n \\
\end{aligned}
\]

◊ Natural separation scheme is still optimal
Mathematical Challenge

- functional optimization problem
- infinite sequence of problems
Mathematical Challenge

- functional optimization problem
- infinite sequence of problems
- “non-convex” constraints
- “non-convex” objective function
Mathematical Challenge

- functional optimization problem
- infinite sequence of problems
- “non-convex” constraints
- “non-convex” objective function

Local calculus-based techniques insufficient
The Proof

- Derivation of a new information-theoretic inequality
- Gaussians have an optimality property
- Proof technique crucially uses:

  operational nature of the problem
Prior Work

Natural Gaussian Separation Scheme
Prior Work

Oohama Outer Bound (1997)
Prior Work

Oohama Outer Bound (1997)

Sufficient to characterize: Sum Rate
Different strategies provide different correlation between errors $(y_1 - \hat{y}_1)$ and $(y_2 - \hat{y}_2)$

Key idea: Form an equivalence class of strategies with the same correlation coefficient
A Natural Lower Bound

- Let θ be the empirical correlation coefficient of the errors.
A Natural Lower Bound

◊ Let $\theta$ be the empirical correlation coefficient of the errors.

◊ Cooperative lower bound:

\[
y_1^n \quad \rightarrow \quad \text{Cooperative Encoder} \quad \rightarrow \quad \hat{y}_1^n \quad \text{Decoder} \quad \rightarrow \quad \hat{y}_2^n
\]

\[
y_2^n \quad \rightarrow \quad \text{Cooperative Encoder} \quad \rightarrow \quad R_1 + R_2 \quad \text{Decoder} \quad \rightarrow \quad \hat{y}_2^n
\]
A Natural Lower Bound

Let $\theta$ be the empirical correlation coefficient of the errors.

Allow encoders to cooperate:

\[ R_1 + R_2 \geq R_{coop}(\theta) := \frac{1}{2} \log \frac{|K_y|}{d_1 d_2 (1 - \theta^2)} \]

Point-to-Point compression problem
Problem of Reconstructing the Sum

\[ y_1^n \rightarrow \text{Enc 1} \rightarrow R_1 \rightarrow \text{Decoder} \rightarrow \hat{y}_\text{sum}^n \]

\[ y_2^n \rightarrow \text{Enc 2} \rightarrow R_2 \rightarrow \text{Decoder} \rightarrow \hat{y}_\text{sum}^n \]

\[ y^n_{\text{sum}} = y_1^n + y_2^n \]
Reconstructing the Sum

\[ y^n_{\text{sum}} = y^n_1 + y^n_2 \]

\[ \frac{1}{n} \sum_{m=1}^{n} E[(y_{\text{sum}}(m) - \hat{y}_{\text{sum}}(m))^2] \leq d_o \]
Reconstructing the Sum

◊ Let $d_o = 2d(1 + \theta)$.

◊ The scheme from before is a feasible scheme for the $y$-sum problem.

◊ Hence

$$R_1 + R_2 \geq R_{\text{sum}}(\theta)$$

◊ Optimal architecture for the $y$-sum problem: **Natural Gaussian Separation Scheme**

  – Oohama, 1997
Two Bounds

Combine the cooperative and $y$-sum lower bounds:

$$R_1 + R_2 \geq \min_{\theta \in (-1,1)} \max (R_{\text{coop}}(\theta), R_{\text{sum}}(\theta))$$
Combine the cooperative and $y$-sum lower bounds:
Two Bounds

- Combine the cooperative and $y$-sum lower bounds:

- Conclusion: $R_{\text{coop}}(\theta^*) \geq R_1 + R_2 \geq R_{\text{coop}}(\theta^*)$
Upshot

◊ Studied the canonical quadratic Gaussian distributed source coding problem

◊ Resolved the optimality of a natural Gaussian analog-digital separation scheme
Mathematical Import

\[
\max \ h(Y_1Y_2|C_1C_2)
\]

- $Y_1, Y_2$ are jointly Gaussian
- $C_1 - Y_1 - Y_2 - C_2$
- $\text{Var}(Y_i|C_1, C_2) \leq d_i, \quad i = 1, 2$
Mathematical Import

\[
\max h(Y_1 Y_2 | C_1 C_2)
\]

- \(Y_1, Y_2\) are jointly Gaussian
- \(C_1 - Y_1 - Y_2 - C_2\)
- \(\text{Var}(Y_i | C_1, C_2) \leq d_i, \quad i = 1, 2\)

Solution:

\((Y_1, Y_2, C_1, C_2)\) jointly Gaussian
Mathematical Import

\[
\min_{Y_1, Y_2} \max_{C_1, C_2} h(Y_1 Y_2 | C_1 C_2) - h(Y_1 Y_2)
\]

\(\diamond\) \(\text{Cov}(Y_1, Y_2)\) is fixed

\(\diamond\) \(C_1 - Y_1 - Y_2 - C_2\)

\(\diamond\) \(\text{Var}(Y_i | C_1, C_2) \leq d_i, \quad i = 1, 2\)

Solution:

\((Y_1, Y_2, C_1, C_2)\) jointly Gaussian
Multi-terminal Information Theoretic Problems

Information Transmission

Information Representation

Multiple Access

Broadcast

Distributed Source Coding

Multiple Description
Multiple Descriptions

Diamond vector Gaussian memoryless source $x^n$
A subset $S$ of descriptions are received, $S \subset \{1, \ldots, L\}$

- Quadratic distortion covariance constraint
  - positive semidefinite partial order
A Classical Open Problem

◊ An achievable scheme
  – El Gamal and Cover, 1982

◊ Optimality for two scalar Gaussian descriptions
  – Ozarow, 1980

◊ Mathematical insight allows us to make progress
Symmetric Multiple Description for Vector Gaussian Source

Either any $k$ descriptions or all descriptions are received
Symmetric Multiple Description for Vector Gaussian Source

Either any $k$ descriptions or all descriptions are received

Main result: Optimality of an analog-digital layered architecture
Mathematical Import

\[
\min \quad I(C_1; C_2; \ldots; C_L) + I(C_1, \ldots, C_L; x^n)
\]

- \(x^n\) i.i.d. Gaussian

- \(\text{Cov}(x^n | C_i) \preceq D_i, \quad i = 1, \ldots, L\)

- \(\text{Cov}(x^n | C_1, \ldots, C_L) \preceq D_0\)
Mathematical Import

\[
\min \ I(C_1; C_2; \ldots; C_L) + I(C_1, \ldots, C_L; \mathbf{x}^n)
\]

- \(\mathbf{x}^n\) i.i.d. Gaussian
- \(\text{Cov}(\mathbf{x}^n|C_i) \preceq \mathbf{D}_i, \quad i = 1, \ldots, L\)
- \(\text{Cov}(\mathbf{x}^n|C_1, \ldots, C_L) \preceq \mathbf{D}_0\)

Solution:

\((\mathbf{x}^n, C_1, \ldots, C_L)\) jointly Gaussian
Mathematical Import

\[
\max_{x^n} \min_{C_1, \ldots, C_L} I(C_1; C_2; \ldots; C_L) + I(C_1, \ldots, C_L; x^n)
\]

- \(x^n\) i.i.d. and \(\text{Cov}(x)\) fixed
- \(\text{Cov}(x^n|C_i) \preceq D_i, \quad i = 1, \ldots, L\)
- \(\text{Cov}(x^n|C_1, \ldots, C_L) \preceq D_0\)

Solution:

\((x^n, C_1, \ldots, C_L)\) jointly Gaussian
Multi-terminal Information Theoretic Problems

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Multiple Description
Vector Gaussian Broadcast Channel

○ Independent messages $W_1, W_2$
Vector Gaussian Broadcast Channel

- Independent messages $W_1, W_2$
- $Z_i^n$ i.i.d. Gaussian $\mathcal{N}(0, K_i)$
Vector Gaussian Broadcast Channel

- Independent messages $W_1, W_2$
- $Z^n_i$ i.i.d. Gaussian $\mathcal{N}(0, \mathbf{K}_i)$
- Quadratic transmit constraint: $\mathbb{E} \left[ \frac{1}{n} \sum_{k=1}^{n} \mathbf{X}_k \mathbf{X}_k^t \right] \preceq \mathbf{S}$
Independent messages $W_1, W_2$

$Z_i^n$ i.i.d. real Gaussian with strictly positive covariance matrix, $i = 1, 2$

Power-covariance constraint: $E\left[\frac{1}{n} \sum_{k=1}^{n} X_k X_k^t\right] \preceq S$

What is the capacity region?
Costa Scheme

\[ W_1^n = X_1^n (W_1) + X_2^n (W_2) + Z_1^n \]

\[ W_2^n = X_1^n (W_1) + X_2^n (W_2) + Z_2^n \]
\[
Y_1^n = X_1^n(W_1) + X_2^n(W_2) + Z_1^n
\]
\[
Y_2^n = X_1^n(W_1) + X_2^n(W_2) + Z_2^n
\]
\[ e_{Y_1^n} = e_{X_1^n}(W_1) + e_{X_2^n}(W_2) + Z_1^n \]

\[ e_{Y_2^n} = e_{X_2^n}(W_2) + Z_2^n \]
Costa Scheme

\[
\begin{align*}
\tilde{Y}_1^n &= \tilde{X}_1^n(W_1) + \tilde{X}_2^n(W_2) + Z_1^n \\
\tilde{Y}_2^n &= \tilde{X}_2^n(W_2) + Z_2^n
\end{align*}
\]
The Converse

- A natural scheme
The Converse

- A natural scheme

- Optimality recently shown
  - Weingarten, Steinberg and Shamai, 2005
The Converse

- A natural scheme

- Optimality recently shown
  - Weingarten, Steinberg and Shamai (2005)

Goal: Identify the underlying information theoretic inequality
The $\mu$-Sum Problem

- A natural scheme

- Optimality recently shown
  - Weingarten, Steinberg and Shamai, 2005

Goal: Identify the underlying information theoretic inequality

- Focus on
  $$\max R_1 + \mu R_2, \quad \text{for } \mu \geq 1$$
Marton Outer Bound

\[(W_1, W_2) \rightarrow \text{Encoder} \rightarrow X^n \rightarrow \sum \rightarrow Y_1^n \rightarrow \hat{W}_1\]

\[(W_1, W_2) \rightarrow \text{Encoder} \rightarrow X^n \rightarrow \sum \rightarrow Y_2^n \rightarrow \hat{W}_2\]
Marton Outer Bound

\[ R_1 \leq I(X; Y_1 | U), \quad R_2 \leq I(U; Y_2) \]
Marton Outer Bound

\( (W_1, W_2) \)

Encoder

\( X^n \)

Decoder 1

\( \hat{W}_1 \)

Decoder 2

\( \hat{W}_2 \)

\( Z^n_1 \)

\( Y^n_1 \)

\( Z^n_2 \)

\( Y^n_2 \)

\( R_1 \leq I(X; Y_1 | U), \quad R_2 \leq I(U; Y_2) \)

\( (X, U) \) are jointly Gaussian.
Maximizing the $\mu$-Sum

\[ R_1 + \mu R_2 \leq I(X; Y_1 | U) + \mu I(U; Y_2) \]

\[ = - h(Z_n^1) + \mu h(X + Z_2) + \sum_u P_U(u) \left[ h(X + Z_1 | U = u) - \mu h(X + Z_2 | U = u) \right] \]
Maximizing the $\mu$-Sum

\[
R_1 + \mu R_2 \leq I(X; Y_1 | U) + \mu I(U; Y_2)
\]
\[
= - h(Z^n_1) + \mu h(X + Z_2)
\]
\[
+ \sum_u P_U(u) [h(X + Z_1 | U = u) - \mu h(X + Z_2 | U = u)]
\]
\[
= - \log(2\pi e)^n |K_{Z_1}|
\]
Maximizing the $\mu$-Sum

$$R_1 + \mu R_2 \leq I(X; Y_1|U) + \mu I(U; Y_2)$$

$$= - h(Z_1^n) - \log(2\pi e)^n |K_{Z_1}|$$

$$+ \mu h(X + Z_2) \leq \frac{\mu}{2} \log(2\pi e)^n |S + K_{Z_2}|$$

$$+ \sum_u P_U(u) [h(X + Z_1|U = u) - \mu h(X + Z_2|U = u)]$$
Maximizing the $\mu$-Sum

\[
R_1 + \mu R_2 \leq I(X; Y_1|U) + \mu I(U; Y_2)
\]
\[
= -h(Z_1^n) = -\log(2\pi e)^n|K_{Z_1}|
\]
\[
+ \mu h(X + Z_2) \leq \frac{\mu}{2} \log(2\pi e)^n|S + K_{Z_2}|
\]
\[
+ \sum_u P_U(u) [h(X + Z_1|U = u) - \mu h(X + Z_2|U = u)]
\]

Tension to be resolved!
Maximizing the $\mu$-Sum

\[ R_1 + \mu R_2 \leq I(X; Y_1|U) + \mu I(U; Y_2) \]
\[ = - h(Z_1^n) = - \log(2\pi e)^n |K_{Z_1}| \]
\[ + \mu h(X + Z_2) \leq \frac{\mu}{2} \log(2\pi e)^n |S + K_{Z_2}| \]
\[ + \sum_u P_U(u) [h(X + Z_1|U = u) - \mu h(X + Z_2|U = u)] \]

\[ \text{Underbrace} \]
\[ \text{Tension to be resolved!} \]

\[ \diamond \text{If} \]
\[ \max_{\text{Cov}(X) \preceq S_u} h(X + Z_1) - \mu h(X + Z_2) \]
\[ \text{is concave in} \ S_u \]
Maximizing the $\mu$-Sum

\[
R_1 + \mu R_2 \leq I(X; Y_1 | U) + \mu I(U; Y_2) = -h(Z_1^n) + \mu h(X + Z_2) + \sum_u P_U(u) [h(X + Z_1 | U = u) - \mu h(X + Z_2 | U = u)]
\]

\[
\leq -\log(2\pi e)^n |K_{Z_1}| + \frac{\mu}{2} \log(2\pi e)^n |S + K_{Z_2}|
\]

Tension to be resolved!

- If

\[
\max_{\text{Cov}(X) \preceq S_u} h(X + Z_1) - \mu h(X + Z_2)
\]

is concave in $S_u$
Main Result

\[
\max_x h(X + Z_1) - \mu h(X + Z_2)
\]

- \(Z_1, Z_2\) Gaussian with strictly positive covariance matrices
- \(X\) independent of \(Z_1, Z_2\)
- \(\text{Cov}(X) \preceq S\)
Main Result

\[
\max_X \ h(X + Z_1) - \mu h(X + Z_2)
\]

- \( Z_1, Z_2 \) Gaussian with strictly positive covariance matrices
- \( X \) independent of \( Z_1, Z_2 \)
- \( \text{Cov}(X) \preceq S \)

Solution:

\( X \) is Gaussian
Corollary 1

\[
\max_{(x_1, x_2)} h(x_1 + x_2)
\]

- \(X_1, X_2\) jointly distributed
- \(\text{Var}(X_1) \leq a_1\)
- \(\text{Var}(X_2) \leq a_2\)
Corollary 1

\[
\max_{(X_1, X_2)} \ h(X_1 + X_2)
\]

- \(X_1, X_2\) jointly distributed
- \(\text{Var}(X_1) \leq a_1\)
- \(\text{Var}(X_2) \leq a_2\)

Solution:

\((X_1, X_2)\) jointly Gaussian and “lined-up”.
Corollary 1

\[
\max_{(X_1, X_2)} h(X_1 + X_2)
\]

\( X_1, X_2 \) jointly distributed

\( h(X_1) \leq a_1 \)

\( h(X_2) \leq a_2 \)
Corollary 1

\[
\max_{(X_1, X_2)} h(X_1 + X_2)
\]

- \(X_1, X_2\) jointly distributed
- \(h(X_1) \leq a_1\)
- \(h(X_2) \leq a_2\)

Solution is not:

\((X_1, X_2)\) jointly Gaussian and “lined-up”

- Cover and Zhang, 1994
Corollary 1

\[
\max_{(X_1, X_2)} h(X_1 + X_2 + Z)
\]

- \( Z \) Gaussian with a strictly positive variance
- \( X_1, X_2 \) jointly distributed
- \( \text{Var}(X_1) \leq a_1 \)
- \( h(X_2 + Z) \leq a_2 \) for \( a_2 \leq a_2^0 \)
Corollary 1

\[
\max_{(X_1, X_2)} h(X_1 + X_2 + Z)
\]

- $Z$ Gaussian with a strictly positive variance
- $X_1, X_2$ jointly distributed
- $\text{Var}(X_1) \leq a_1$
- $h(X_2 + Z) \leq a_2$ for $a_2 \leq a_2^o$

Solution:

$(X_1, X_2)$ jointly Gaussian and “lined-up”
Corollary 2

\[
\min_X h(X + Z)
\]

- \( Z \) Gaussian with a strictly positive covariance matrix
- \( X \) independent of \( Z \)
- \( h(X) \geq a \)
Corollary 2

$$\min_{X} h(X + Z)$$

- $Z$ Gaussian with a strictly positive covariance matrix
- $X$ independent of $Z$
- $h(X) \geq a$

Entropy Power Inequality:

- $X$ Gaussian
- covariance matrix proportional to $K_Z$
Corollary 2

$$\min_X h(X + Z)$$

- $Z$ Gaussian with a strictly positive covariance matrix
- $X$ independent of $Z$
- $h(X) \geq a$
- $\text{Cov}(X) \preceq S$
Corollary 2

\[
\min_X h(X + Z)
\]

- Z Gaussian with a strictly positive covariance matrix
- X independent of Z
- \( h(X) \geq a \)
- \( \text{Cov}(X) \preceq S \)

Solution:

\( X \) is still Gaussian.
An Isoperimetric View

\[ \min_X h(X + Z) \]

- \( Z \) Gaussian with a strictly positive covariance matrix
- \( X \) independent of \( Z \)
- \( h(X) \geq a \)
An Isoperimetric View

\[
\min_{\mathbf{X}} \mu h(\mathbf{X} + \mathbf{Z}) - h(\mathbf{X})
\]

- \( \mathbf{Z} \) Gaussian with a strictly positive covariance matrix
- \( \mathbf{X} \) independent of \( \mathbf{Z} \)
- \( \mu \geq 1 \)
An Isoperimetric View

\[
\min_{X} \mu h(X + Z) - h(X)
\]

- A monotone path (Dembo, Cover and Thomas, 1991)

\[X_\lambda := \sqrt{1 - \lambda}X_0 + \sqrt{\lambda}X_G^*\]

\[X_0: \text{ arbitrary}\]

Classical Fisher Information Inequality
An Isoperimetric View

\[
\min_X h(X + Z)
\]

- \( Z \) Gaussian with a strictly positive covariance matrix
- \( X \) independent of \( Z \)
- \( h(X) \geq a \)
- \( \text{Cov}(X) \preceq S \)
An Isoperimetric View

\[ \min_{\boldsymbol{X}} \mu h(\boldsymbol{X} + \boldsymbol{Z}) - h(\boldsymbol{X}) \]

- \( \boldsymbol{Z} \) Gaussian with a strictly positive covariance matrix
- \( \boldsymbol{X} \) independent of \( \boldsymbol{Z} \)
- \( \mu \geq 1 \)
- \( \text{Cov}(\boldsymbol{X}) \preceq \boldsymbol{S} \)
An Isoperimetric View

$$\min_x \mu h(X + Z) - h(X)$$

- A monotone path

$$X_0 : \text{Cov}(X_0) \preceq S$$

$$X^*_G : \text{optimal Gaussian solution}$$

$$X_\lambda := \sqrt{1 - \lambda} X_0 + \sqrt{\lambda} X^*_G$$

Novel Fisher Information Inequality
Credits

◊ Distributed Source Coding
  – Aaron Wagner and Saurabh Tavildar
  – arXiv:cs.IT/0510095

◊ Multiple Descriptions
  – Hua Wang
  – arXiv:cs.IT/0510078

◊ Broadcast Channel
  – Tie Liu
  – arXiv:cs.IT/0604025