The Rate Region of the Quadratic Gaussian Two-Terminal Source-Coding Problem

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The Problem
The Problem

\[ y_1^n \rightarrow \text{Enc 1} \rightarrow R_1 \rightarrow \hat{y}_1^n \]

\[ y_2^n \rightarrow \text{Enc 2} \rightarrow R_2 \rightarrow \hat{y}_2^n \]

\[ [y_1(m), y_2(m)] \text{ i.i.d. real Gaussian with covariance matrix } K_y \]
The Problem

- $[y_1(m), y_2(m)]$ i.i.d. real Gaussian with covariance matrix $K_y$

- Quadratic distortion constraints

\[
\frac{1}{n} \sum_{m=1}^{n} E[(y_i(m) - \hat{y}_i(m))^2] \leq d_i, \text{ for } i = 1, 2
\]
The Problem

- \([y_1(m), y_2(m)]\) i.i.d. real Gaussian with covariance matrix \(K_y\)
- Quadratic distortion constraints
  \[
  \frac{1}{n} \sum_{m=1}^{n} E[(y_i(m) - \hat{y}_i(m))^2] \leq d_i, \quad i = 1, 2
  \]

What is the rate-distortion region?
Lossless Distributed Compression

- Distortions tolerated: $d_1 = d_2 = 0$

- Classical result of Slepian and Wolf:
  a binning scheme ensures no-loss distributed compression
Lossless Distributed Compression

- Distortions tolerated: $d_1 = d_2 = 0$

- Classical result of Slepian and Wolf:
  
  A binning scheme ensures no-loss distributed compression

- Suggests a natural strategy for lossy distributed compression:
  
  - First, digitize the analog source into bits
  
  - Second, use Slepian-Wolf binning to convey the bits losslessly
A Natural Separation Scheme

$y_1^n \xrightarrow{\text{VQ–1}} \text{Binning Scheme–1} \xrightarrow{R_1} \hat{y}_1^n$

$y_2^n \xrightarrow{\text{VQ–2}} \text{Binning Scheme–2} \xrightarrow{R_2} \hat{y}_2^n$
A Natural Gaussian Separation Scheme

\[ y_1^n \xrightarrow{VQ-1} u_1^n \xrightarrow{Binning Scheme-1} R_1 \rightarrow \hat{y}_1^n \]

\[ y_2^n \xrightarrow{VQ-2} u_2^n \xrightarrow{Binning Scheme-2} R_2 \rightarrow \hat{y}_2^n \]

\[ \diamond \text{ VQ-1 and VQ-2 are Gaussian quantizers} \]
A Natural Gaussian Separation Scheme

VQ-1 and VQ-2 are Gaussian quantizers

Is this scheme optimal?
Main Result

- The Gaussian separation scheme is optimal for Gaussian sources
Main Result

- The Gaussian separation scheme is \textit{optimal} for Gaussian sources.

- Further, Gaussian statistics is the \textit{worst-case} scenario for this scheme.
Prior Work

Natural Gaussian Separation Scheme
Prior Work

Oohama Outer Bound (1997)
Prior Work

Oohama Outer Bound (1997)

To be characterized: Sum Rate
A Natural Lower Bound

- Let $\theta$ be the empirical correlation coefficient of the errors.
A Natural Lower Bound

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- Cooperative lower bound:
A Natural Lower Bound

◊ Let $\theta$ be the empirical correlation coefficient of the errors.

◊ Cooperative lower bound:

$$R_1 + R_2 \geq R_{\text{coop}}(\theta) := \frac{1}{2} \log \frac{|K_y|}{d_1 d_2 (1 - \theta^2)}$$
The $\mu$-sum Problem

$y_1^n \rightarrow \text{Enc 1} \rightarrow R_1 \rightarrow \text{Decoder} \rightarrow \hat{y}_\mu^n$

$y_2^n \rightarrow \text{Enc 2} \rightarrow R_2 \rightarrow \text{Decoder} \rightarrow \hat{y}_\mu^n$

$y_\mu^n = \mu_1 y_1^n + \mu_2 y_2^n$
The $\mu$-sum Problem

\[ y^n_\mu = \mu_1 y^n_1 + \mu_2 y^n_2 \]

\[ \frac{1}{n} \sum_{m=1}^{n} E[(y_\mu(m) - \hat{y}_\mu(m))^2] \leq d_\mu \]
The $\mu$-sum Problem

Let $d_\mu = \mu_1^2 d_1 + \mu_2^2 d_2 + 2\mu_1 \mu_2 \sqrt{d_1 d_2} \theta$.

The scheme from before is a feasible scheme for the $\mu$-sum problem.

Hence

$$R_1 + R_2 \geq R_{\mu\text{-sum}}(\theta)$$
The CEO and $\mu$-sum Problems
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*For $\mu_1 \cdot \mu_2 > 0$, the $\mu$-sum problem is equivalent to a quadratic Gaussian CEO problem.*
Natural Gaussian Separation Scheme is Optimal
Natural Gaussian Separation Scheme is Optimal

- Oohama 1997, 2003
Optimal Natural Gaussian Separation scheme

\[ y_1^n \xrightarrow{\text{VQ–1}} u_1^n \xrightarrow{\text{Binning Scheme–1}} R_1 \xrightarrow{\text{Decoder}} \hat{y}_1^n \]

\[ y_2^n \xrightarrow{\text{VQ–2}} u_2^n \xrightarrow{\text{Binning Scheme–2}} R_2 \xrightarrow{\text{Decoder}} \hat{y}_2^n \]
Let $\theta^*$ be the correlation coefficient for the optimal natural Gaussian separation scheme for the original problem.
Extracting Common Randomness

◊ The **optimal** natural Gaussian scheme is also optimal for the $\mu$-sum problem with

\[
\begin{align*}
\mu_1 &= \sqrt{d_2} \\
\mu_2 &= \sqrt{d_1} \\
d_\mu &= \mu_1^2 d_1 + \mu_2^2 d_2 + 2\mu_1 \mu_2 \sqrt{d_1 d_2} \theta^* \end{align*}
\]
The Lower Bound

Combine the cooperative and $\mu$-sum lower bounds:

\[ R_1 + R_2 \geq \min_{\theta \in (-1,1)} \max (R_{\text{coop}}(\theta), R_{\mu-\text{sum}}(\theta)) \]
The Lower Bound

○ Combine the cooperative and $\mu$-sum lower bounds:

$$R_1 + R_2 \geq \min_{\theta \in (-1,1)} \max (R_{\text{coop}}(\theta), R_{\mu-\text{sum}}(\theta))$$

○ We know $R_{\text{coop}}(\theta^*) = R_{\mu-\text{sum}}(\theta^*)$
The Lower Bound: Visualized

$R_{\text{coop}}(\theta)$ and $R_{\text{sum}}(\theta)$
The Lower Bound: Visualized

\[ R_{\text{coop}}(\theta^*) \geq R_1 + R_2 \geq R_{\text{coop}}(\theta^*) \]
Mathematical Import

\[
\max h(Y_1Y_2|C_1C_2)
\]

- \(Y_1, Y_2\) are jointly Gaussian
- \(C_1 - Y_1 - Y_2 - C_2\)
- \(\text{Var}(Y_i|C_1, C_2) \leq d_i, \quad i = 1, 2\)
Mathematical Import

\[
\max h(Y_1Y_2|C_1C_2)
\]

○ \( Y_1, Y_2 \) are jointly Gaussian

○ \( C_1 - Y_1 - Y_2 - C_2 \)

○ \( \text{Var}(Y_i|C_1, C_2) \leq d_i, \quad i = 1, 2 \)

Solution:

\[(Y_1, Y_2, C_1, C_2) \quad \text{jointly Gaussian}\]
\[ \min_{Y_1, Y_2} \max_{C_1, C_2} \ h(Y_1 Y_2 | C_1 C_2) - h(Y_1 Y_2) \]

\[ \diamond \ \text{Cov}(Y_1, Y_2) \text{ is fixed} \]

\[ \diamond \ C_1 - Y_1 - Y_2 - C_2 \]

\[ \diamond \ \text{Var}(Y_i | C_1, C_2) \leq d_i, \quad i = 1, 2 \]

Solution:

\((Y_1, Y_2, C_1, C_2) \text{ jointly Gaussian}\)
An Extension: Multiple \( \mu \)-sums Problem

\[ y_1^n = \mu_{11} y_1^n + \mu_{12} y_2^n \]
\[ y_2^n = \mu_{21} y_1^n + \mu_{22} y_2^n \]
Remote Terminal Distributed Source Coding

Symmetric version of the problem proposed by Oohama, 2005
Natural Gaussian Separation Scheme is Optimal
Conclusion

- Studied the canonical quadratic Gaussian distributed source coding problem
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- Resolved the optimality of a natural Gaussian analog-digital separation scheme
Conclusion

◊ Studied the canonical quadratic Gaussian distributed source coding problem

◊ Resolved the optimality of a natural Gaussian analog-digital separation scheme

◊ Credit: Aaron Wagner and Saurabh Tavildar

◊ arXiv:cs.IT/0510095