Opportunistic Orthogonal Writing on Dirty Paper

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The Dirty Paper Channel

\[ y(t) = x(t) + s(t) + w(t) \]

WGN

p.s.d. \( \frac{N_0}{2} \)
The Dirty Paper Channel

\[ y(t) = x(t) + s(t) + w(t) \]

WG Interference: p.s.d. \( \frac{N_s}{2} \)

WGN: p.s.d. \( \frac{N_0}{2} \)
The Dirty Paper Channel

\[ y(t) = x(t) + s(t) + w(t) \]

- **Power** $P$
- **WG Interference**
- **WGN**

- **Bandwidth** $W$
  - p.s.d. $\frac{N_s}{2}$
  - p.s.d. $\frac{N_0}{2}$

- Observation (Cos81): Capacity of the channel is

\[ W \log \left( 1 + \frac{P}{N_0 W} \right) \]

- An abstract **binning** scheme
Focus of this talk

\[ y(t) = x(t) + s(t) + w(t) \]

- Power \( P \)
- Wideband
- WG Interference
- WGN
- p.s.d. \( \frac{N_s}{2} \)
- p.s.d. \( \frac{N_0}{2} \)

◊ Wideband dirty paper channel
Main Result

- Setting: Wideband channel
  - Criterion: Minimum Energy per reliable Bit $\mathcal{E}_b$

Main Result:

- An explicit binning scheme
- Ramifications:
  * Typical error events
  * Low rate VQ for wideband Gaussian source
  * Generalization to abstract alphabets
PPM and Wideband AWGN Channel

$M$ Pulses

- Each message corresponds to a pulse
Opportunistic PPM

$K$ Sub-pulses

$M$ Pulses

- Each message corresponds to $K$ sub-pulses
Encoding Opportunistic PPM

◊ Transmit sub-pulse (among $K$ possible choices) where $s(t)$ is largest
Decoding Opportunistic PPM

Choose sub-pulse (among $MK$ possible choices) where $y(t)$ is largest
Relieving the Tension

- Transmit signal amplitude

\[ \sqrt{\mathcal{E}_s} = \sqrt{\mathcal{E}_b \log M} + \sqrt{N_s \log K} \]

- \( \sqrt{N_s \log K} \) is the opportunistic amplitude gain

- Larger \( K \) means larger signal amplitude

- But decoder has more choices to get confused (\( MK \) sub-pulses)
Relieving the Tension

- Transmit signal amplitude
  \[ \sqrt{\mathcal{E}_s} = \sqrt{\mathcal{E}_b \log M} + \sqrt{N_s \log K} \]

- Error probability
  \[ \mathbb{P} [\text{Error}] \leq MK \quad \mathbb{P} [s + w \geq \sqrt{\mathcal{E}_s}] \]
Relieving the Tension

◊ Transmit signal amplitude

\[ \sqrt{\mathcal{E}_s} = \sqrt{\mathcal{E}_b \log M} + \sqrt{N_s \log K} \]

◊ Error probability

\[
\mathbb{P} \left[ \text{Error} \right] \leq MK \mathbb{P} \left[ s + w \geq \sqrt{\mathcal{E}_s} \right]
\]

\[
\approx MK \mathbb{P} \left[ s \geq \frac{N_s}{N_s + N_0} \sqrt{\mathcal{E}_s} \right] \mathbb{P} \left[ w \geq \frac{N_0}{N_s + N_0} \sqrt{\mathcal{E}_s} \right]
\]
Relieving the Tension

◊ Transmit signal amplitude

\[ \sqrt{\mathcal{E}_s} = \sqrt{\mathcal{E}_b \log M} + \sqrt{N_s \log K} \]

◊ Error probability

\[ \mathbb{P} \left[ \text{Error} \right] \leq MK \mathbb{P} \left[ s + w \geq \sqrt{\mathcal{E}_s} \right] \]

\[ \approx MK \mathbb{P} \left[ s \geq \frac{N_s}{N_s + N_0} \sqrt{\mathcal{E}_s} \right] \mathbb{P} \left[ w \geq \frac{N_0}{N_s + N_0} \sqrt{\mathcal{E}_s} \right] \]

\[ \approx M \mathbb{P} \left[ w \geq \sqrt{\mathcal{E}_b \log M} \right] \]

◊ If possible, then desired result follows
Choice of $K$

- Transmit signal amplitude
  \[ \sqrt{\mathcal{E}_s} = \sqrt{\mathcal{E}_b \log M} + \sqrt{N_s \log K} \]

- Choose $\log K = \frac{N_s}{N_0} \log M$

- Then
  \[ \frac{N_s}{N_s + N_0} \sqrt{\mathcal{E}_s} = \sqrt{N_s \log K}, \quad \frac{N_0}{N_s + N_0} \sqrt{\mathcal{E}_s} = \sqrt{\mathcal{E}_b \log M} \]
Tension at correct choice of $K$

- Transmit signal amplitude $\sqrt{\mathcal{E}_s} = \sqrt{\mathcal{E}_b \log M} + \sqrt{N_s \log K}$

- Choose $\log K = \frac{N_s}{N_0} \log M$

$$\frac{N_s}{N_s + N_0} \sqrt{\mathcal{E}_s} = \sqrt{N_s \log K}, \quad \frac{N_0}{N_s + N_0} \sqrt{\mathcal{E}_s} = \sqrt{\mathcal{E}_b \log M}$$

- Error probability

$$\mathbb{P} [\text{Error}] \leq MK \mathbb{P} \left[ s \geq \frac{N_s}{N_s + N_0} \sqrt{\mathcal{E}_s} \right] \mathbb{P} \left[ w \geq \frac{N_0}{N_s + N_0} \sqrt{\mathcal{E}_s} \right]$$

$$= MK \mathbb{P} \left[ s \geq \sqrt{N_s \log K} \right] \mathbb{P} \left[ w \geq \sqrt{\mathcal{E}_b \log M} \right]$$
The AWGN Performance

◊ Transmit signal amplitude

\[ \sqrt{E_s} = \sqrt{E_b \log M} + \sqrt{N_s \log K} \]

◊ Choose

\[ \log K = \frac{N_s}{N_0} \log M \]

\[ \frac{N_s}{N_s + N_0} \sqrt{E_s} = \sqrt{N_s \log K}, \quad \frac{N_0}{N_s + N_0} \sqrt{E_s} = \sqrt{E_b \log M} \]

◊ Error probability

\[ \mathbb{P}[\text{Error}] \leq MK \mathbb{P} \left[ s \geq \frac{N_s}{N_s + N_0} \sqrt{E_s} \right] \mathbb{P} \left[ w \geq \frac{N_0}{N_s + N_0} \sqrt{E_s} \right] \]

\[ = MK \mathbb{P} \left[ s \geq \sqrt{N_s \log K} \right] \mathbb{P} \left[ w \geq \sqrt{E_b \log M} \right] \]

\[ \approx M \mathbb{P} \left[ w \geq \sqrt{E_b \log M} \right] \]

◊ AWGN performance is achieved
Non-Gaussian Dirt

- Interference $s(t)$ is not AWG
- Make length $N$ of each pulse long enough:
  \[
  \frac{\sum_{i=1}^{N} s_i}{\sqrt{N}} \sim \mathcal{N}
  \]
- Essentially convert dirt into Gaussian
- **Robustness**: OPPM works over a (restricted) arbitrarily varying dirt
Story So Far

- OPPM is a simple concrete scheme to achieve AWGN performance over wideband Dirty Paper channel
Pedagogical Utility

- OPPM is a simple concrete scheme to achieve AWGN performance over wideband Dirty Paper channel

  A text book example
Utility of OPPM

- OPPM is a simple concrete scheme to achieve AWGN performance over wideband Dirty Paper channel

  A text book example

- OPPM is also useful to prove new results
Typical Error Events

diamond Error occurs in two ways:
Typical Error Events

- Error occurs in two ways:
  - Opportunistic gain is not large enough

\[
\mathbb{P} \left[ \max_{k=1 \ldots K} s_k < \sqrt{N_s \log K} \right]
\]
Typical Error Events

- Error occurs in two ways:
  - Opportunistic gain is not large enough
    \[ P \left( \max_{k=1 \ldots K} s_k < \sqrt{N_s \log K} \right) \]
  - Interference plus Noise is too large
    \[ P \left[ s + w > \sqrt{\mathcal{E}_s} \right] \]
Typical Error Events

Diamond Error occurs in two ways:

Diamond Opportunistic gain is not large enough

\[
P \left[ \max_{k=1\ldots K} s_k < \sqrt{N_s \log K} \right] \approx \exp \left( \exp \left( -\epsilon_1 \log K \right) \right)
\]

Diamond Interference plus Noise is too large

\[
P \left[ s + w > \sqrt{\mathcal{E}_s} \right] \approx \exp \left( -\epsilon_2 \log K \right)
\]
Typical Error Events

- Error occurs in two ways:
  - **Opportunistic gain is not large enough**
    \[ P \left[ \max_{k=1 \ldots K} s_k < \sqrt{N_s \log K} \right] \approx \exp \left( -\exp (\epsilon_1 \log K) \right) \]
  - **Interference plus Noise is too large**
    \[ P \left[ s + w > \sqrt{\mathcal{E}_s} \right] \approx \exp (-\epsilon_2 \log K) \]
  - **Typical error event same as in wideband AWGN channel**

Error exponent of wideband dirty paper channel same as that of AWGN channel
OPPM is Structured Binning

- Interpretation of Costa’s binning scheme
  - Each bin is a good quantizer
  - All bins put together, good channel code
OPPM is Structured Binning

◊ Interpretation of Costa’s binning scheme
  – Each bin is a good quantizer
  – All bins put together, good channel code

◊ OPPM fits this theme
  – Overall it is PPM - a good channel code

◊ Where is the quantizer?
Pulse Position Quantization

Source

Reconstruction
Optimality of PPQ

- Consider performance criterion:

\[
\max \frac{\text{Quantization Rate}}{\text{Reduction in Distortion}}
\]
Optimality of PPQ

◦ Consider performance criterion:

$$\max \frac{\text{Quantization Rate}}{\text{Reduction in Distortion}}$$

◦ An optimality property:

PPQ is a best low rate VQ of a wideband Gaussian source
A Deeper Understanding

◊ Opportunistic PPM is

a combination of **PPM** (a good wideband channel code)
and **PPQ** (a good wideband VQ)
A Deeper Understanding

◊ Opportunistic PPM is

   a combination of PPM (a good wideband channel code)
   and PPQ (a good wideband VQ)

◊ Generalization:

   – Abstract alphabet Gelfand-Pinsker channel
   – Abstract cost measure over input alphabet
   – Performance measure:

     Capacity per unit cost of an abstract Gelfand-Pinsker channel
Gelfand-Pinsker Channel

△ Memoryless Channel

△ Alphabets: $x \in \mathcal{X}$, $s \in \mathcal{S}$, $y \in \mathcal{Y}$

△ Cost: $b(x)$ for $x \in \mathcal{X}$
Capacity per Unit Cost

- Largest rate of reliable communication per unit cost:

\[
\max_{P_{XU|S}} \frac{I(U; Y) - I(U; S)}{\mathbb{E}[b(X)]}.
\]

- \( U \) is an auxiliary random variable

- Achieved by an abstract binning scheme
Capacity per Unit Cost

- Largest rate of reliable communication per unit cost:
  \[
  \max_{P_{XU|S}} \frac{I(U;Y) - I(U;S)}{\mathbb{E}[b(X)]}.
  \]

- \(U\) is an auxiliary random variable

- Achieved by an abstract binning scheme

- Main result:
  
  Generalized Opportunistic PPM achieves the capacity per unit cost

- Need a zero cost input letter
Generalized PPM

N is length of each pulse

M Pulses

Achieves capacity per unit cost of a classical channel

Generalized Opportunistic PPM

Each message corresponds to $K$ sub-pulses
Encoding

- Transmit $x = f(s)$ over sub-pulse that has type $V$
- Deterministic function $f : S \rightarrow \mathcal{X}$
Encoding

- Transmit $x = f(s)$ over sub-pulse that has type $V$
- Deterministic function $f : S \rightarrow \mathcal{X}$
- Need $K = \exp(D(V||S))$
Decoding

- **Binary Hypothesis Test** at any sub-pulse

\[ \mathcal{H}_0 : \mathbb{P}_{Y|X} = f(V), V \]

\[ \mathcal{H}_1 : \mathbb{P}_{Y|X} = 0, S \]
Decoding

- **Binary Hypothesis Test** at any sub-pulse
  
  \[ \mathcal{H}_0 : \mathbb{P}_{Y|X=f(V),V} \]
  
  \[ \mathcal{H}_1 : \mathbb{P}_{Y|X=0,S} \]

- Fix \( \mathbb{P}[\mathcal{H}_0 \rightarrow \mathcal{H}_1] = \epsilon \).

- **Stein’s Lemma (almost)**
  
  \[ \mathbb{P}[\mathcal{H}_1 \rightarrow \mathcal{H}_0] = \exp \left( -N \, D \left( \mathbb{P}_{Y|X=f(V),V} \| \mathbb{P}_{Y|X=0,S} \right) \right) \]
Decoding

◊ *Binary Hypothesis Test* at any sub-pulse

\[ \mathcal{H}_0 : \mathbb{P}_Y|X=f(V),V \]

\[ \mathcal{H}_1 : \mathbb{P}_Y|X=0,S \]

◊ Fix \( \mathbb{P} [\mathcal{H}_0 \rightarrow \mathcal{H}_1] = \epsilon. \)

◊ Stein’s Lemma (almost)

\[
\mathbb{P} [\mathcal{H}_1 \rightarrow \mathcal{H}_0] = \exp(-N D (\mathbb{P}_Y|X=f(V),V||\mathbb{P}_Y|X=0,S))
\]

◊ Independent tests for each sub-pulse

– MK tests
Rate per Unit Cost

◊ Error Probability

\[ P[\text{Error}] \leq M \ K \ \exp\left(-N \ D\left(\mathbb{P}_Y|X=f(V),V||\mathbb{P}_Y|X=0,S\right)\right) \]

\[ = M \ \exp\left(N \ D\left(V||S\right)\right) \ \exp\left(-N \ D\left(\mathbb{P}_Y|X=f(V),V||\mathbb{P}_Y|X=0,S\right)\right) \]

\[ = M \ \exp\left(-N \ \left(D\left(\mathbb{P}_Y|X=f(V),V||\mathbb{P}_Y|X=0,S\right) - D\left(V||S\right)\right)\right) \]
Rate per Unit Cost

◊ Error Probability

\[
P[\text{Error}] \leq M \ K \ \exp\left( -N \ D \left( \mathbb{P}_{Y|X=f(V),V} || \mathbb{P}_{Y|X=0,S} \right) \right) \\
= M \ \exp\left( -N \ D \left( \mathbb{P}_{Y|X=f(V),V} || \mathbb{P}_{Y|X=0,S} \right) - D \left( V || S \right) \right)
\]

◊ Cost incurred

\[
N \ \mathbb{E} \left[ b\left( f(V) \right) \right]
\]
Rate per Unit Cost

- Error Probability

\[ \mathbb{P} [\text{Error}] \leq M K \exp \left( -N D \left( \mathbb{P}_Y | X = f(V), V \middle| \mathbb{P}_Y | X = 0, S \right) \right) = M \exp \left( -N \left( D \left( \mathbb{P}_Y | X = f(V), V \middle| \mathbb{P}_Y | X = 0, S \right) - D (V \middle| S) \right) \right) \]

- Cost incurred

\[ N \mathbb{E} [b(f(V))] \]

- Conclusion: achievable rate per unit cost

\[ \frac{\log M}{N \mathbb{E} [b(f(V))]} = \frac{D \left( \mathbb{P}_Y | X = f(V), V \middle| \mathbb{P}_Y | X = 0, S \right) - D (V \middle| S)}{\mathbb{E} [b(f(V))]} \]
**Main Result**

- **Theorem**: The capacity per unit cost is

\[
\max_{\mathbb{P}_V \ll \mathbb{P}_S, \ f: S \to \mathcal{X}} D \left( \mathbb{P}_{Y|X=f(V), V} \| \mathbb{P}_{Y|X=0, S} \right) - D (V \| S) - \mathbb{E} \left[ b(f(V)) \right]
\]

- Converse follows by calculus on

\[
\max_{\mathbb{P}_{XU|S}} \frac{I(U; Y) - I(U; S)}{\mathbb{E} \left[ b(X) \right]}
\]

- Key idea: identify

\[ V \] (the preferred type) from

\[ U \] (the auxiliary random variable)
Hindsight

◊ Return to Dirty Paper Channel:

\[ y[m] = x[m] + s[m] + w[m] \]

◊ Cost function \( b(x) = x^2 \)

◊ Choose \( X = x \) and \( V = \frac{N_s}{N_0} x + S \)

◊ Then

\[
\frac{D \left( \mathbb{P}_{Y \mid X=x, V} \mid \mathbb{P}_{Y \mid X=0, S} \right) - D \left( V \mid S \right)}{x^2} = \frac{1}{N_0}
\]
A Summary

- Capacity of Gelfand-Pinsker Channels achieved by binning scheme

- Binning scheme
  - Each bin a good quantizer
  - Overall good channel code

- Consider cost-efficient quantization and channel coding
  - S. Verdú, “Capacity per Unit Cost”

- Combination of cost-efficient quantizers and channel codes

  achieves capacity per unit cost of Gelfand-Pinsker channel
Another Ramification

- Consider low-rate VQ of wideband Gaussian source $x_1(t)$
- Decoder has access to correlated wideband Gaussian source $x_2(t)$
- A type of Wyner-Ziv lossy compression
  - A binning scheme suggests no “rate loss”
- Ideas developed here suggest an explicit scheme: OPPQ
OPPQ: Encoding

Source

Reconstruction without side information
OPPQ: Reconstruction

Source

Reconstruction with side information