Approximately Optimal Wireless Broadcasting

Sreeram Kannan, Adnan Raja, and Pramod Viswanath

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Abstract

We study a wireless broadcast network, where a single source reliably communicates independent messages to multiple destinations, with the aid of relays and cooperation between destinations. The wireless nature of the medium is captured by the *broadcast* nature of transmissions as well as the *superposition* of all transmit signals plus independent Gaussian noise at the received signal at any radio. We propose a scheme that can achieve rate tuples within a constant gap away from the cut-set bound, where the constant is independent of channel coefficients and power constraints.

The proposed scheme operates in two steps. The *inner* code, in which the relays perform a quantize-and-encode operation, is constructed by lifting a scheme designed for a corresponding discrete superposition network. The *outer* code is a Marton code for the non-Gaussian vector broadcast channel induced by the relaying scheme, and is constructed by adopting a "*receiver-centric*" viewpoint.

1 Introduction

The scenario of study in this paper is depicted in Figure 1. A single source node is reliably communicating independent messages to multiple destination nodes using the help of multiple relay nodes. In the example of a cellular system, the setting represents downlink communication where the base-station is transmitting to multiple terminals with the potential help of relay stations. Note that some of the terminals can themselves act as relays. A special instance of our setting is the following: only the source node and multiple destinations (i.e., no relays) are present. Since we have allowed the ability to transmit and receive at all nodes, this special instance models the downlink of a cellular system with the destinations having the ability to cooperate among themselves, which has been studied in [9].

We consider the canonical Gaussian channel model among the various nodes in the network: time is discrete and synchronized among all the nodes. Denoting the baseband transmit symbol (a complex number) of node k at time m by $x_k[m]$, the average transmit power constraint at each node implies that,

$$\sum_{m=1}^{T} |x_k[m]|^2 \leq TP_k, \tag{1}$$

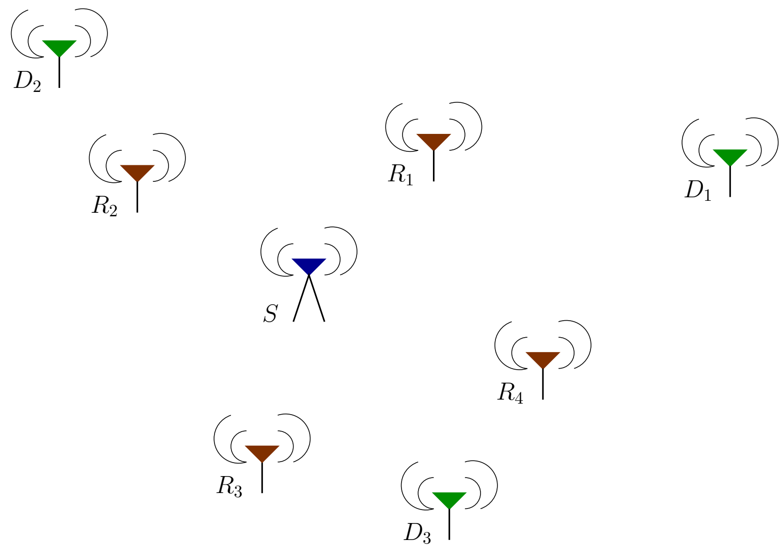


Figure 1: A wireless broadcast network

where T is the time period over which the communication occurs. At each time m, we have the received signal at any node ℓ

$$y_\ell[m] = \sum_{k \neq \ell} h_{k\ell}[m] x_k[m] + z_\ell[m]. \tag{2}$$

Here $\{z_\ell[m]\}_m$ is i.i.d. Gaussian noise and independent across the different nodes ℓ. The channel attenuation $h_{k\ell}$ between a pair of nodes (k, ℓ) is supposed to be constant over the time scale of communication. Note that by normalizing the channel attenuation $h_{k\ell}$, without loss of generality, we will assume unit average power constraints at each node, i.e. $P_k = 1$ and also the variance of $z_\ell[m]$ to be 1. We suppose full duplex mode of operation for the most part, while discussing the implications of half duplex mode later in the paper. We suppose single antenna at each node and leave the discussion with multiple antennas for a later part of the paper. We will begin with the supposition that these channel attenuations are known to all the nodes in the network, and revisit this requirement later.

Let $\mathcal{V}$ denote the set of all nodes in the network. A $(2^{TR_1}, \ldots, 2^{TR_J}, T)$ coding scheme for the broadcast network, with source node S and destination nodes $D_1, \ldots, D_J$, which communicates over T time instants is comprised of the following

1. Independent random variables W_i which are distributed uniformly on $[2^{TR_i}]$ for $i = 1, \ldots, J$ respectively. W_i denotes the message intended for destination D_i.

2. The source mapping,
$$f_S : (W_1 \times \ldots \times W_J) \to \mathcal{X}_S^T. \tag{3}$$

3. The relay mappings for each $v \in \mathcal{V} \setminus \{S\}$ and $t \in [T]$,
$$f_{v,t} : \mathcal{Y}_v^{t-1} \to \mathcal{X}_v. \tag{4}$$

4. The decoding map at destination D_i,

$$g_{D_i} : \mathcal{Y}_{D_i}^T \to \hat{W}_i. \tag{5}$$

The probability of error for destination i under this coding scheme is given by

$$P_e^i \stackrel{\text{def}}{=} \Pr\{\hat{W}_i \neq W_i\}. \tag{6}$$

A rate tuple $(R_1, R_2, ..., R_J)$, where R_i is the rate of communication in bits per unit time for destination D_i, is said to be achievable if for any $\epsilon > 0$, there exists a $(2^{TR_1}, 2^{TR_2}, ..., 2^{TR_J}, T)$ scheme that achieves a probability of error lesser than ϵ for all nodes, i.e., $\max_i P_e^i \leq \epsilon$. The capacity region $\mathcal{C}$ is the set of all achievable rates.

The following is the well known cut-set upper bound to the rate tuples of reliable communication [5, 4]: denoting the set of all nodes by $\mathcal{V}$; and for all subsets $\mathcal{J} \subseteq [J]$, where $[J]$ denotes the set $\{1, \dots, J\}$, denoting $\Lambda_{\mathcal{J}}$ to be the collection of all subsets $\Omega \subset \mathcal{V}$ such that the source nodes $S \in \Omega$ and a subset $\mathcal{J} \subseteq [1 : J]$ of destinations $\mathcal{D}_{\mathcal{J}} \in \Omega^c$; we have that if $(R_1, \dots, R_J)$ is achievable, $\forall \mathcal{J}$, there is a joint distribution $p(\{X_v | v \in \mathcal{V}\})$ (denoted by Q) such that

$$R_{\mathcal{J}} \leq \bar{C}_{\mathcal{J}}(Q) \stackrel{\text{def}}{=} \min_{\Omega \in \Lambda_{\mathcal{J}}} I\left(Y_{\Omega^c}; X_{\Omega} | X_{\Omega^c}\right), \tag{7}$$

where $R_{\mathcal{J}} \stackrel{\text{def}}{=} \sum_{j \in \mathcal{J}} R_j$.

Let $\bar{\mathcal{C}}(Q)$ denote the set of all rate tuples that satisfy the cut-set upper bound for a given joint distribution Q, and $\bar{\mathcal{C}}$ denote the cut-set bound:

$$\bar{\mathcal{C}}(Q) \stackrel{\text{def}}{=} \{(R_1, ..., R_J) : R_{\mathcal{J}} \leq \bar{C}_{\mathcal{J}}(Q) \ \forall \mathcal{J} \subseteq [1 : J]\} \tag{8}$$

$$\bar{\mathcal{C}} \stackrel{\text{def}}{=} \text{conv}\left(\bigcup_{\{Q : \mathbb{E}|X_v|^2 \leq 1\}} \bar{\mathcal{C}}(Q) \right), \tag{9}$$

where $\text{conv}(\cdot)$ denotes the convex hull of the region.

Our main result is the following.

Theorem 1. *For the wireless broadcast network, a rate vector $(R_1, \dots, R_J)$ is achievable if $\forall\ \mathcal{J}$,*

$$(R_1 + k, \dots, R_J + k) \in \bar{\mathcal{C}} \tag{10}$$

for some constant k, which depends only on the number of nodes, and not on the channel coefficients, and $k = O(|\mathcal{V}| \log |\mathcal{V}|)$.

With a single destination node ($J = 1$), the scenario reduces to the classical Gaussian (unicast) relay channel. Pioneering work of [1] has obtained an approximate characterization of the capacity for this scenario. In recent work, [3, 6] derive similar approximation results with different coding schemes. In particular, [1, 3] take a two-step approach in their coding scheme: first, they develop deterministic models that approximate the Gaussian channel and next, they construct the codes for the Gaussian channel based on the codes for the corresponding deterministic channel approximation. The specific approaches adopted in the actual deterministic approximation and moving from the codes for the deterministic channel to the Gaussian one are different between [1] and [3]; as such, we follow the approach of [3] in our proof of the main result.

The proposed scheme operates in two steps. The *inner* code, in which the relays essentially perform a quantize-and-encode operation, is constructed by lifting a scheme designed for a corresponding discrete superposition network. This induces a vector broadcast channel between

The *outer* code is essentially a Marton code ([7, 8]) for the broadcast channel induced by the relaying scheme, and is constructed based on a "receiver-centric" viewpoint.

The rest of the paper is organized as follows. In Section 2, we give a coding scheme and establish an achievable rate region for deterministic broadcast networks. In Section 3, we prove Theorem 1 by giving a coding scheme for the wireless broadcast network. In order to do so, we use the discrete superposition network, which is an approximation to the wireless network, and the "lift" the scheme from the discrete superposition network to the Gaussian network. In Section 4, we discuss various aspects of the proposed scheme, primarily the reciprocity in the context of linear deterministic and Gaussian networks, and the channel state information required. In Section 5, various generalizations of the scheme are provided, for half-duplex networks, for networks with multiple antenna and for broadcast wireless networks, where some set of nodes demand the same information and other nodes demand independent information.

2 Deterministic Broadcast Networks

In the deterministic network model, the received signal at each node is a deterministic function of the received signals.

$$y_\ell[m] = g_\ell\left(\{x_k[m]\}_{k\neq\ell}\right). \tag{11}$$

The input and output alphabet sets, $\mathcal{X}_k$'s and $\mathcal{Y}_\ell$'s respectively, are assumed to be finite sets.

As before, we have the following cut-set upper bound to the rate tuples of reliable communication [5, 4]: if $(R_1, \ldots, R_J)$ is achievable, then $\forall \mathcal{J} \subseteq [1 : J]$, there is a joint distribution $p\left(\{X_v | v \in \mathcal{V}\}\right)$ (denoted by Q) such that

$$R_{\mathcal{J}} \leq \bar{C}_{\mathcal{J}}(Q) \stackrel{\text{def}}{=} \min_{\Omega\in\Lambda_{\mathcal{J}}} I\left(Y_{\Omega^c}; X_\Omega | X_{\Omega^c}\right). \tag{12}$$

We prove the following achievability result for the deterministic channel.

Theorem 2. *For the deterministic broadcast network, a rate vector $(R_1, \ldots, R_J)$ is achievable if $\forall$ $\mathcal{J}$, there is a product distribution $\prod_{v\in\mathcal{V}} p(X_v)$ (denoted by Q_p) such that*

$$R_{\mathcal{J}} \leq \bar{C}_{\mathcal{J}}(Q_p). \tag{13}$$

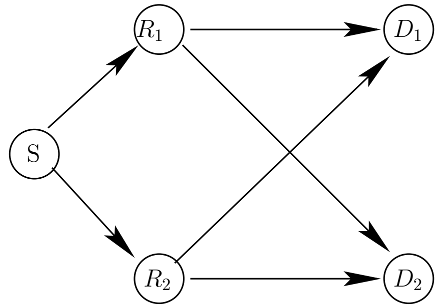


Figure 2: A Layered broadcast-relay network

Remark 1. *Aref networks, i.e., deterministic broadcast networks where each node can receive information from every incoming edge separately, were studied in [24]. It is shown there that cut-set bound can be achieved for these networks. This result can also be recovered from Theorem 2 by observing that product form distributions optimize the cut-set bound. It should be noted, however, that the scheme proposed in [24] is a separation-based scheme whereas the scheme proposed here is not.*

Proof. We prove Theorem 2 for the layered network here. The arguments can then be extended to the general network by using time expansion as done in [1]. A network is called a *L-layered network* if the set of vertices $\mathcal{V}$ can be partitioned into L disjoint sets, such that only the source node S is in the first layer and the J destination nodes are in the L-th layer. The nodes in the intermediate layers are relaying nodes. The received signal at the nodes in the $l+1$-th layer only depend on the transmitted signals at the nodes in the l-th layer. This dependency is often represented by edges connecting the nodes from the l-th layer to the $(l+1)$-th layer. An example of a layered broadcast network is shown in Fig. 2. The advantage of working with a layered network is that we can view the information as propagating from one layer to the next without getting intertwined.

2.1 Outline of Coding Scheme

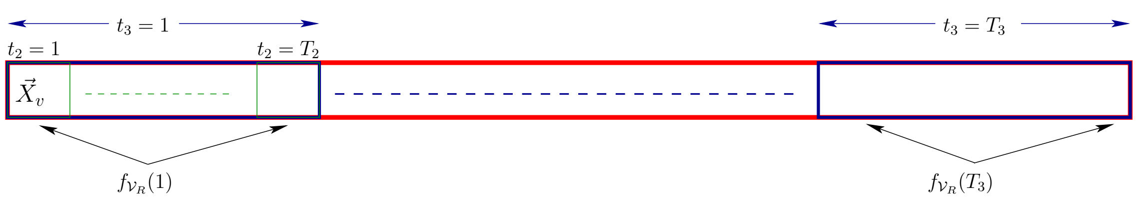


Figure 3: Coding scheme

The coding scheme operates over three levels of nested blocks as shown in Figure 3. If T is the total time period of communication, then $T = T_1T_2T_3$. The innermost level is level-1 and consists of T_1 time instants. T_2 such level-1 blocks constitutes a level-2 block. The overall coding scheme operates over T_3 level-2 blocks. Our coding scheme comprises of the following:

- The *message* W_i is broken into T_3 independent sub-messages $W_i(1), \ldots, W_i(T_3)$, where each sub-message is encoded over a particular level-2 block.

- The *relay mappings* are done at the level-1 block. Every node blocks up T_1 received symbols and maps it to T_1 transmit symbols. This relay mapping is fixed for the duration of a level-2 block and it essentially creates an end-to-end deterministic broadcast channel at the level-2 block.

- The *encoding at the source* is done on the induced end-to-end deterministic broadcast channel for each level-2 block. Across different level-2 blocks, the mappings are generated in an i.i.d. manner and corresponding broadcast schemes are used for the different end-to-end broadcast channel induced by this operation. This is very much like a fading broadcast channel. Here the random fading is introduced by the relay mappings. This is merely a proof technique which allows us to average the performance over random relay mappings.

- The *destinations* decode the sub-message corresponding to each level-2 independently and sequentially.

2.2 Coding Scheme in Detail

Throughout the discussion below, we fix a particular product distribution Q_p, which is then used to describe a random ensemble of coding operations.

2.2.1 Relay mappings

In the proposed scheme, the relays operate over level-1 blocks, i.e., each relay transmits in a level-1 block using only the information from the previous received level-1 block. We will use $\vec{x}_r \stackrel{\text{def}}{=} x_r^{T_1}$ and $\vec{y}_r \stackrel{\text{def}}{=} y_r^{T_1}$ to denote the transmit and receive block at any relay node $r \in \mathcal{V}_R$, where $\mathcal{V}_R$ is the set of all relay nodes. The mapping at the relay node denoted by

$$f_r(t_3) : \mathcal{Y}_r^{T_1} \to \mathcal{X}_r^{T_1}, \tag{14}$$

is random and generated i.i.d. from the distribution $p(X_r)$, i.e., $\forall y_r^{T_1} \in \mathcal{Y}_r^{T_1}$, generate $x_r^{T_1}$ i.i.d. from $p(X_r)$. Note that the relay mapping is only a function of t_3 i.e., it is fixed across all level-1 blocks in a given level-2 block. Across different level-2 blocks, i.e., for each t_3, it is generated in an i.i.d. manner. As mentioned earlier, this induces an end-to-end broadcast channel as shown in Figure 4; a different one across every level-2 block.

2.2.2 Source Mappings

The capacity of the deterministic broadcast channel is well known ([7, 10]). For our coding scheme over the induced deterministic broadcast channel of Figure 4, we use the coding scheme similar to the one described for the deterministic broadcast channel in [8], which we refer to henceforth as the "Marton code". The *source codebook* for the t_3-th level-2 block, which maps

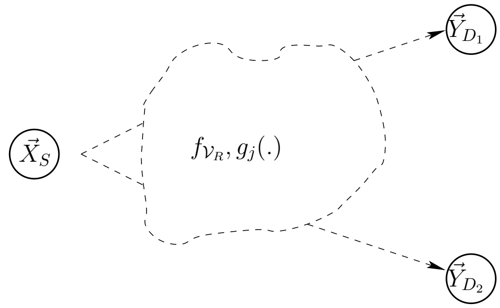


Figure 4: Effective end-2-end deterministic broadcast channel created by a level-1 code.

the message $W_i(t_3) \in [2^{T_1 T_2 R_i(t_3)}], i = 1, 2, ..., J$ to transmit symbol block $\vec{x}_S^{T_2}$, is described below.

Given the random vector $\vec{X}_S$ which is distributed as $p(\vec{X}_S) = \prod p(X_S)$, the channel and the relay mapping induce the joint distribution over the random variables $\left(\vec{X}_S, \vec{Y}_{\mathcal{V}}\right)$. Create auxiliary random variables $\vec{U}_{D_i}$ such that $p_{\vec{X}, \vec{U}_{D_1}, \vec{U}_{D_2}, ..., \vec{U}_{D_J}}$ is the same as $p_{\vec{X}, \vec{Y}_{D_1}, \vec{Y}_{D_2}, ..., \vec{Y}_{D_J}}$.

The set $\mathcal{T}_\delta^{T_2}(\vec{U}_{D_i})$ of all typical $\vec{u}_{D_i}^{T_2}$ are binned into $2^{T_1 T_2 R_i(t_3)}$ bins, where each bin index corresponds to a message, for $i = 1, 2, ..., J$. For each vector $(\vec{u}_{D_1}^{T_2}, \ldots, \vec{u}_{D_J}^{T_2}) \in \mathcal{T}_\delta^{T_2}(\vec{Y}_{D_1}, \ldots, \vec{Y}_{D_J})$, there exists a sequence $\vec{x}_S^{T_2}(\vec{u}_{D_1}^{T_2}, \ldots, \vec{u}_{D_J}^{T_2})$, since the channel is deterministic, such that $(\vec{x}_S^{T_2}, \vec{u}_{D_1}^{T_2}, \ldots, \vec{u}_{D_J}^{T_2}) \in \mathcal{T}_\delta^{T_2}(\vec{X}_S, \vec{Y}_{D_1}, \ldots, \vec{Y}_{D_J})$. This specifies the codebook for the given level-2 block t_3. Similar codebooks are generated, statistically independently, for all the T_3 level-2 blocks.

2.2.3 Encoding

The messages $W_i \in [2^{T_1 T_2 T_3 R_i}]$ are split into sub-messages $W_i(t_3) \in [2^{T_1 T_2 R_i(t_3)}]$ such that

$$\sum_{t_3} R_i(t_3) = R_i T_3. \tag{15}$$

For the t_3-th level-2 block, the messages to be transmitted are given by $(W_1(t_3), \ldots, W_J(t_3))$ for the J destinations respectively. To transmit the message, the source looks at the codebook for the level-2 block t_3 and tries to find a vector $\left(\vec{u}_1^{T_2}, \ldots, \vec{u}_J^{T_2}\right) \in \mathcal{T}_\delta^{T_2}(\vec{U}_1, \ldots, \vec{U}_J)$ such that $\vec{u}_i^{T_2}$ is also in the bin with index $W_i(t_3)$. If the source can find such a vector, it transmits $\vec{x}_s^{T_2}(\vec{u}_1^{T_2}, \ldots, \vec{u}_J^{T_2})$. If the source cannot find such a sequence it transmits a random sequence.

2.2.4 Decoding

At the end of t_3-th level-2 block, the destination D_i decodes the transmitted message $W_i(t_3)$. The destination tries to find the bin in which the received level-2 block $\vec{y}_{D_i}^{T_2}(t_3)$ falls and decodes that bin index as the transmitted message.

2.3 Performance Analysis

We begin with identifying rate constraints for the t_3-th level-2 block, so that arbitrarily small probability of error can be achieved for decoding the message at this block. From the coding theorem for the deterministic broadcast channel ([7], Theorem 3), we know that as long as $R_i(t_3)$ satisfies, for $i = 1, \ldots, J$ and $\forall \mathcal{J} \subseteq \{1, \ldots, J\}$,

$$R_{\mathcal{J}}(t_3) \leq \frac{1}{T_1} H(\vec{Y}_{D_{\mathcal{J}}} | F_{\mathcal{V}_{\mathcal{R}}} = f_{\mathcal{V}_{\mathcal{R}}}(t_3)), \tag{16}$$

the probability of error can be made arbitrarily small by choosing a large enough T_2. The overall rate R_i is given by (15). Therefore the rate tuple $(R_1, \ldots, R_J)$ satisfies

$$R_{\mathcal{J}} \leq \frac{1}{T_1 T_3} \sum_{t_3=1}^{T_3} H(\vec{Y}_{D_{\mathcal{J}}} | F_{\mathcal{V}_{\mathcal{R}}} = f_{\mathcal{V}_{\mathcal{R}}}(t_3)), \forall\, \mathcal{J} \subseteq \{1, \ldots, J\}\,. \tag{17}$$

By the strong law of large numbers, as $T_3 \to \infty$, we have

$$\begin{aligned} \frac{1}{T_1 T_3} \sum_{t_3=1}^{T_3} H(\vec{Y}_{D_{\mathcal{J}}} | F_{\mathcal{V}_{\mathcal{R}}} = f_{\mathcal{V}_{\mathcal{R}}}(t_3)) &\overset{\text{a.s.}}{\to} \frac{1}{T_1} \mathbb{E} H(\vec{Y}_{D_{\mathcal{J}}} | F_{\mathcal{V}_{\mathcal{R}}} = f_{\mathcal{V}_{\mathcal{R}}}(t_3)) \\ &= \frac{1}{T_1} H(\vec{Y}_{D_{\mathcal{J}}} | F_{\mathcal{V}_{\mathcal{R}}}). \end{aligned} \tag{18}$$

Next, we relate the expression in (18) to the cut-sets using the following lemma:

Lemma 1. *Given arbitrary* $\epsilon > 0$, $\exists\, T_1$ *s.t.,*

$$\begin{aligned} H(\vec{Y}_{D_{\mathcal{J}}} | F_{\mathcal{V}_R}) = H(Y^{T_1}_{D_{\mathcal{J}}} | F_{\mathcal{V}_R}) \geq T_1(\bar{C}_{\mathcal{J}}(Q_p) - \epsilon), \\ \forall\, \mathcal{J} \subseteq \{1, \ldots, J\}\,. \end{aligned} \tag{19}$$

Proof. See Appendix A. □

Using (17), (18) and Lemma 1, we conclude that for any rate tuple satisfying

$$R_{\mathcal{J}} \leq \bar{C}_{\mathcal{J}}(Q_p) - \epsilon, \tag{20}$$

arbitrarily small probability of error is achieved. The proof is finally completed by allowing time-sharing between the coding schemes for all product distributions Q_p. □

3 Gaussian Broadcast Networks (Proof of Theorem 1)

3.1 Layered Network

As for the deterministic network, we will first consider a layered network. For the deterministic network, we first used the random relay mappings to create an effective vector broadcast channel. Then we used the Marton scheme with a specific choice of auxiliary random variables, which is optimal for the deterministic broadcast channel. For the Gaussian network, while it is possible to do the inner code similarly and induce a broadcast channel, this is a vector non-Gaussian broadcast channel for which it is unknown whether a Marton scheme can achieve any rate within a constant gap of the cut-set bound. To deal with this issue, we convert the Gaussian network into a deterministic network, for which we can design the germane code and then appropriately "lift" the code from the deterministic to the Gaussian network.

The procedure is outlined as follows:

1. Given the Gaussian broadcast network, we construct a corresponding deterministic superposition network (DSN). The cut-set bound of the DSN approximates the cut-set bound of the corresponding Gaussian network to within a factor $N \log N$. Further, the DSN is deterministic and thus the scheme in Theorem 2 can be used for the DSN to achieve the cut-set bound evaluated under product-form distributions. The details are provided in Sec 3.1.1.

2. We then prune a natural coding scheme $\mathcal{P}(\kappa)$ for the DSN, such that the rate is reduced by a factor $N\kappa$. The details are in Sec. 3.1.2.

3. Finally, we show that for an appropriate choice of $\kappa = O(\log N)$, the pruned coding scheme on the DSN can be *emulated* on the Gaussian network, because each node in the Gaussian network can decode the corresponding received vector in the DSN. The details are given in Sec. 3.1.3. Therefore, the rate achieved in the Gaussian network by the proposed scheme achieves within a gap of $O(N \log N)$ of the cut-set bound.

3.1.1 Discrete Superposition Network (DSN)

Given a Gaussian network, we construct a discrete superposition network. This network is essentially a truncated noiseless version of the Gaussian model. Further, the input in this model is restricted to a finite set.

Corresponding to the channel model for the Gaussian network given by (2), the received signal in the DSN is given by

$$y_\ell[m] = \left[\sum_{k \neq \ell} h_{k\ell}[m] x_k[m] \right], \tag{21}$$

where $[\cdot]$ lies in $\mathbb{Z}+\imath\mathbb{Z}$ and corresponds to rounding the real and imaginary parts of the complex number to the nearest integer. The transmit symbols $x_k[m]$ are restricted to a discrete and finite complex valued set. This defines the DSN. Note that our model is very similar to the truncated model in [1] and to the model described in [3].

Next, the following lemma relates the cut-set upper bound of the Gaussian network, $\bar{\mathcal{C}}^{\text{Gauss}}$, to the cut-set of the DSN under product distribution.

Lemma 2. *There exists a Q_p for the DSN such that*

$$\bar{\mathcal{C}}^{Gauss} \subseteq \bar{\mathcal{C}}^{DSN}(Q_p) + k_2(1, 1, ..., 1), \tag{22}$$

where $k_2 = O(|\mathcal{V}|)$.

Proof. See Appendix D. □

Note that since the DSN is a deterministic network, we have the following corollary of Theorem 2.

Corollary 1. *For the DSN, a rate vector* $(R_1, \ldots, R_J)$ *is achievable if there is a product distribution* Q_p *such that,* $\forall\ \mathcal{J}$*,*

$$R_{\mathcal{J}} \leq \bar{C}^{DSN}_{\mathcal{J}}(Q_p). \tag{23}$$

We observe that, here Q_p is product distribution over the *finite* input alphabet set of the DSN.

3.1.2 Pruned Coding Scheme $\mathcal{P}(\kappa)$ for the DSN

The pruned coding scheme described next is along the lines of the ideas developed in [3]. The main idea there is that a coding scheme from the DSN can be used in the Gaussian network, if at every node, the received vector at any node in the DSN can be decoded from the corresponding received vector in the Gaussian network. This will not, in general, be true for any scheme in the DSN. Therefore, to get the source codebook for the Gaussian network, the codewords transmitted by the source in the DSN are pruned to a smaller set in such a way that the received vector at any node in the DSN can be decoded from the received vector in the Gaussian network. In this section, we will demonstrate how to construct a pruned coding scheme $\mathcal{P}(\kappa)$, for which the rate is reduced by a constant $N\kappa$. In Sec. 3.1.3, we will show how to choose the parameter κ such that the Gaussian network can *emulate* the DSN.

Our coding scheme for the deterministic network described in Section 2.2 involved three levels. Level-1 involved relay mappings over a level-1 block of T_1 time symbols defined by $f_{\mathcal{V}_R}$. Then the level-2 code involved a Marton scheme for this fixed $f_{\mathcal{V}_R}$. This code was over a level-2 block, which comprised of T_2 level-1 blocks. Then at level-3, we repeated this Marton code over T_3 level-2 blocks, where for each super-block the relay mapping $f_{\mathcal{V}_R}$ was generated i.i.d. using $F_{\mathcal{V}}$. For the Gaussian network, we will maintain the *same level-*1 *and level-*3 *codes*, but will modify the Marton code by pruning the set of codewords transmitted by the source. This is described in detail next.

*Pruning the Level-*2 *(Marton) code:* We now specify the operation during the t_3-th level-2 block. In this level-2 block we need to send the messages $W_i \in [2^{T_1 T_2 R_i(t_3)}]$ at rates $R_i(t_3)$, for $i = 1, \ldots, J$. As before, since the the relays maintain the same mapping $f_{\mathcal{V}_R}(t_3)$ for the duration of this level-2 block, we have an effective end-to-end deterministic channel as shown in Figure 4.

As before, given the random variable at the source $\vec{X}_S = X_S^{T_1}$ which is distributed as $p(\vec{X}_S) = \prod p(X_S)$, the fixed relay mapping and the channel induces the joint distribution on $(\vec{X}_S, \vec{Y}_{\mathcal{V}})$. Let $\mathcal{T}_\delta^{T_2}(\vec{Y}_v)$ denote the set of all typical received block vectors at node v.

Previously, we created the codebook by binning the typical sets $\mathcal{T}_\delta^{T_2}(\vec{Y}_{D_i})$. But now, we prune this set before binning. This will, in effect, lead to pruning the set of typical vectors $\mathcal{T}_\delta^{T_2}(\vec{Y}_v)$ at all nodes. To do this pruning, we fix a constant κ, which will be defined in

Section 3.1.3. At every node v, we pick a random $2^{-T_1 T_2 \kappa}$ fraction of $\mathcal{T}_\delta^{T_2}(\vec{Y}_v)$ and call this subset $\mathfrak{S}_v$. We then define,

$$\mathfrak{Z}_v \stackrel{\text{def}}{=} \{\vec{y}_v^{T_2} : \forall i,\ \exists \vec{y}_i^{T_2} \in \mathfrak{S}_i,\ \text{s.t. } \vec{y}_{\mathcal{V}}^{T_2} \in \mathcal{T}_\delta^{T_2}(\vec{Y}_{\mathcal{V}})\}. \tag{24}$$

We will prune our codebook so that only codewords in $\mathfrak{Z}_v$ are typically received.

Following the Marton code for the deterministic channel, we bin the set of all $\mathfrak{Z}_{D_i}$, the typically received vectors at the destination, into $2^{T_1 T_2 R_i(t_3)}$ bins, for $i = 1, \ldots, J$. For a given message pair $W_1, ..., W_J$, we select a jointly typical $\vec{y}_{D_1}, ..., \vec{y}_{D_J} \in \mathcal{T}_\delta^{T_2}(\mathfrak{Z}_{D_1}, ..., \mathfrak{Z}_{D_J})$ with $\vec{y}_{D_i}$ belonging to the bin corresponding to the message W_i, and transmit the $\vec{x}^{T_2}$ which is jointly typical with all the $\vec{y}_{D_i}$. The decoder on receiving a $\vec{y}_{D_i}$, finds the index of the bin into which it falls and reports this as the message W_i. This is our pruned codebook for the level-2 block t_3. Similarly pruned codebooks are generated for all the T_3 level-2 blocks.

Performance Analysis:

As before, we first find the rate constraints for the t_3-th level-2 block, so that arbitrarily small probability of error can be achieved for decoding the message at this block. For this purpose, we first need to establish a lower bound on the size of $\mathfrak{Z}_{D_i}$. Towards this, we have the following lemma.

Lemma 3.

$$|\mathfrak{Z}_v| \ \dot{>} \ 2^{T_2(H(\vec{Y}_v|F_{\mathcal{V}_R}=f_{\mathcal{V}_R})-NT_1\kappa)} \ \textit{w.h.p. as } T_2 \to \infty, \tag{25}$$

where $N = |\mathcal{V}| - 1$.

We use the notation $a_n \dot{>} 2^{nb}$, to denote the existence of a nonnegative sequence $\epsilon_n \to 0$ such that $a_n > 2^{n(b-\epsilon_n)}$. Notations $\dot{<}$ and $\doteq$ are used in a similar sense in the rest of the paper. For a sequence of events $\mathfrak{E}(n)$ indexed by n, we use "$\mathfrak{E}(n)$ w.h.p. as $n \to \infty$" to denote that $\mathbb{P}\{\mathfrak{E}(n)\} \to 1,$ as $n \to \infty$. In this section T_2 will play the role of n.

Proof. The details of the proof are in Appendix B. To prove this lemma, we use the randomness in the choice of the pruned subsets $\mathfrak{S}_v$. □

The following lemma then characterizes the achievable rates, $R_i(t_3)$, by our pruned scheme in a t_3-th level-2 block.

Lemma 4. *As long as* $R_i(t_3)$ *satisfies,* $\forall i = 1, \ldots, J, \forall \mathcal{J} \subseteq \{1, \ldots, J\}$,

$$R_{\mathcal{J}}(t_3) \leq \frac{1}{T_1} H(\vec{Y}_{D_{\mathcal{J}}}|F_{\mathcal{V}_{\mathcal{R}}} = f_{\mathcal{V}_{\mathcal{R}}}(t_3)) - |\mathcal{J}|N\kappa, \tag{26}$$

the probability of error can be made arbitrarily small by choosing a large enough T_2.

Proof. The result follows from the fact that, our pruned version of the Marton scheme bins $\mathfrak{Z}_{D_i}$ instead of $\mathcal{T}_\delta^{T_2}(\vec{Y}_{D_i})$. The details of the proof are in Appendix C. □

As before, we can now average over the rates in each level-2 block, to get the overall rate R_i. Under the pruned scheme, each rate is reduced by $N\kappa$. Thereby, we have the following lemma.

Lemma 5. *A rate tuple $R = (R_1, \ldots, R_J)$ is achievable for the DSN, using the pruned scheme $\mathcal{P}(\kappa)$, if $\forall\ \mathcal{J}$, there is a product distribution $\prod_{v\in\mathcal{V}} p(X_v)$ (denoted by Q_p) such that*

$$R_{\mathcal{J}} \leq \bar{C}_{\mathcal{J}}(Q_p) - |\mathcal{J}|N\kappa. \tag{27}$$

3.1.3 Code for the Gaussian Network

We will now show that the pruned scheme for the DSN can be lifted to a scheme for the Gaussian network to establish the following result.

Lemma 6. *If $\kappa = \log(12N-2)+11$, then any rate tuple $(R_1, R_2, ..., R_J)$ that can be achieved in the DSN using the pruned coding scheme $\mathcal{P}(\kappa)$ can also be achieved in the Gaussian network.*

Proof. Note that the pruned coding scheme $\mathcal{P}(\kappa)$ for the DSN operated over three levels of nested blocks. This scheme is now adapted to the Gaussian network as follows. We will use $\underline{X}_v, \underline{Y}_v$ to denote the level-2 blocks in the Gaussian network (corresponding to $\vec{X}_v, \vec{Y}_v$ in the DSN).

1. The coding over level-2 blocks is performed in the same way as in the DSN. We will describe the operation during the t_3-th level-2 block. In this level-2 block the relay mapping is fixed to $f_{\mathcal{V}_R}(t_3)$, we will suppress this conditioning henceforth for notational convenience.

2. Given a message $W_i(t_3)$, the source transmits the same vector that it would have transmitted in the DSN, i.e., $\underline{x}_S^{T_2}(W_1(t_3), ..., W_J(t_3)) = \vec{x}_S^{T_2}(W_1(t_3), ..., W_J(t_3))$.

3. Node v receives $\underline{y}_v^{T_2}$ and decodes $\vec{y}_v^{T_2}$, the corresponding vector that it would have received in the DSN.

   Lemma 7. *If $\kappa = \log(12N-2)+11$, then the probability of error for decoding the received vector $\vec{y}_v^{T_2}$ in the DSN from the corresponding vector $\underline{y}_v^{T_2}$ in the Gaussian network goes to zero as $T_2 \to \infty$.*

   Proof. There is a natural joint distribution on $(\vec{X}_S, \vec{Y}_v, \underline{Y}_v)$ given by

$$\begin{aligned} p(\vec{X}_S, \vec{Y}_v, \underline{Y}_v) &= p(\vec{X}_S)p(\vec{Y}_v|\vec{X}_S)p(\underline{Y}_v|\vec{X}_S) &(28)\\ &= p(\vec{X}_S)p(\vec{Y}_v|\vec{X}_S)p(\underline{Y}_v|\underline{X}_S). &(29)\end{aligned}$$

   Here $p(\vec{Y}_v|\vec{X}_S)$ and $p(\underline{Y}_v|\underline{X}_S)$ are a function of the relay mappings. The relay node in the Gaussian network finds the $\vec{y}_v^{T_2} \in \mathfrak{S}_v$ such that[1]

$$(\underline{y}_v^{T_2}, \vec{y}_v^{T_2}) \in \mathcal{T}_\delta^{T_2}(\underline{Y}_v, \vec{Y}_v). \tag{30}$$

[1]While the relay also knows that the received vector should be contained in $\mathfrak{Z}_v \subseteq \mathfrak{S}_v$, we do not explicitly use this information in the decoding step.

$\mathfrak{S}_v$ is a random $2^{-T_1T_2\kappa}$ fraction of the typical set $\mathcal{T}_\delta^{T_2}(\vec{Y}_v)$. Therefore

$$|\mathfrak{S}_v| \doteq 2^{-T_2T_1\kappa}2^{T_2H(\vec{Y}_v)}. \tag{31}$$

If we choose $\kappa = \log(12N-2)+11$, then $T_1\kappa > H(\vec{Y}_v|\underline{Y}_v)$ (see [3] for a similar argument). Therefore,

$$|\mathfrak{S}_v| \dot{<} 2^{-T_2H(\vec{Y}_v|\underline{Y}_v)}2^{T_2H(\vec{Y}_v)} \tag{32}$$

$$\Rightarrow |\mathfrak{S}_v| \dot{<} 2^{T_2(I(\vec{Y}_v;\underline{Y}_v))}. \tag{33}$$

This condition ensures that the typical set decoding succeeds with high probability, and thus we can decode $\vec{Y}_v^{T_2}$ from $\underline{Y}_v^{T_2}$. □

4. Any node v now performs the same mapping as in the DSN,

$$\vec{x}_v^{T_2} = f_v(\vec{y}_v^{T_2}), \tag{34}$$

and transmits the vector $\underline{x}_v^{T_2} = \vec{x}_v^{T_2}$.

5. The destination D_i reconstructs the corresponding vector $\vec{y}_{D_i}^{T_2}$ from $\underline{y}_{D_i}^{T_2}$, and since decoding is possible in the DSN, it is possible in the Gaussian network as well.

To conclude: this procedure yields a code for the Gaussian network that can achieve the same reliable rate of communication as the pruned code in the DSN. □

3.1.4 A Simplified Coding Scheme

Our coding scheme for the deterministic and the Gaussian networks comprised coding over three levels of blocks. We will show now that, if we consider the maximization of a linear functional of the rate, the third level is not necessary (i.e., $T_3 = 1$ is sufficient) in an operational scheme, and is used only as a random coding technique. Suppose we want to maximize a certain linear functional of the rate,

$$\max_{R\in\bar{\mathcal{C}}(Q_p)} \sum_i \lambda_i R_i. \tag{35}$$

In Section 2.2, we saw that the average of the rate across various level-2 blocks yields the rate $R_i = \frac{1}{T_3}\sum_{t_3=1}^{T_3} R_i(t_3)$. Thus

$$\sum_i \lambda_i R_i = \frac{1}{T_3}\sum_{t_3=1}^{T_3}\sum_i \lambda_i R_i(t_3), \tag{36}$$

which implies that there exists a t_3 such that $\sum_i \lambda_i R_i(t_3) \geq \sum_i \lambda_i R_i$. If we use the corresponding relay encoding functions $f_{\mathcal{V}_R}(t_3)$ throughout, then we achieve the best possible linear functional of the rate under this scheme. Thus coding over T_3 level-2 blocks is unnecessary for maximizing a linear functional in the rate region. Thus coding over multiple relay transformations is only a proof technique and not a method of operation of the network for maximizing a linear functional of the rate.

By the convexity of the rate region of the scheme, all extreme points in the rate region can be achieved by setting $T_3 = 1$. To achieve any point inside the rate region, time sharing of these schemes will be required in general. Since the relay encoding functions $f_{\mathcal{V}_R}(t_3)$ required to achieve this may be different for different extreme points, time sharing can be thought of as being equivalent to coding with a larger T_3 .

3.2 Non-layered Networks

Consider a general network specified by the set of vertices $\mathcal{V}$. In [18, 1], it has been shown in the context of unicast traffic that any network can be unfolded in time to get a layered network. In the broadcast scenario here, we use the same procedure. The layering strategy is briefly described below. Similar to [1], the level-1 (inner) block is now over KT_1 time symbols. The relay node still does random mappings over blocks of T_1 symbols, however the transmit vector in k-th block, for $k = 1, \ldots, K$, now depends on the last $k - 1$ received blocks. This relaying scheme can then be represented as a layered-network in time. The induced layered network then has K layers, and each layer has $|\mathcal{V}| + J + 1$ nodes - the $|\mathcal{V}|$ nodes $v[k], k = 1, 2, ..., |\mathcal{V}|$ corresponding to the network at time slot k and $J + 1$ special nodes $T[k], R_1[k], ..., R_J[k]$ that act as virtual transmitters and receivers that act as transmit and receive buffers, holding all the information about source message in the case of transmit buffer and holding all the received information in the case of received buffer. The set of edges connecting the adjacent layers is derived from the network by the following procedure:

1. $v_1[k]$ is connected to $v_2[k + 1]$ with a link h_{v_1,v_2} which is the channel corresponding to the link from v_1 to v_2 in the original network.

2. Memory inside a node is maintained by connecting $v_i[k]$ to $v_i[k+1]$ using an orthogonal and infinite capacity link.

3. The transmit buffer is maintained by connecting $T[k]$ to $T[k+1]$ and also $T[k]$ to $S[k+1]$ using orthogonal and infinite capacity links.

4. Receive buffer for the the destination node j is maintained by connecting $R_j[k]$ to $R_j[k+1]$ and from $D_j[k]$ to $D_j[k + 1]$ using orthogonal and infinite capacity links.

5. A 0-th layer with $S = T[0]$ alone, and a $K + 1$-th layer with $D_j = R[k + 1], j = 1, ..., J$ are added to serve as the source and destinations in this unfolded network.

Let us consider the cut-set bound (normalized by K) between the source and the destination nodes $\mathcal{J}$ for this unfolded network $\bar{C}_{\mathcal{J}}^{K-\text{unf}}(Q)$ and for the original network by $\bar{C}_{\mathcal{J}}^{\text{org}}(Q)$. Then, we have the following lemma (See proof of Lemma 5.1 in [1]).

Lemma 8. $\bar{C}_{\mathcal{J}}^{K-unf}(Q) = \frac{K-|\mathcal{V}|}{K}\bar{C}_{\mathcal{J}}^{org}(Q)$.

If we take K large enough, the effect of the penalty term $\frac{K-|\mathcal{V}|}{K}$ can be made as small as desired. Lemma 8, combined with our result for layered networks, yields the desired result for general (non-layered) networks.

4 Discussion

In this section, we first review linear deterministic broadcast networks, and the capacity-achieving schemes for such networks. We then show how these schemes are reciprocal to the schemes for the multi-source single-destination deterministic networks. Then we proceed to identify the intuition connecting the reciprocal schemes based on a contrast between *transmitter-centric* and *receiver-centric* viewpoints, and use this intuition to point out the inspiration for the Gaussian broadcast network scheme. At the end of this section, we point out the channel state information required at various nodes in order to implement these schemes.

4.1 Linear Deterministic Networks (LDN)

A deterministic network of particular interest is the linear finite-field broadcast deterministic network [1]. The inputs and outputs are vectors over a finite field, i.e. $\mathcal{X}_j = \mathcal{Y}_j = \mathbb{F}_p^q$, for some prime p and $q \in \mathbb{N}$. The channels are linear transformations over this finite field i.e.,

$$y_j[m] = \sum_{i \in N_j} G_{i,j} x_i[m], \tag{37}$$

where $G_{i,j} \in \mathbb{F}_p^{q \times q}$. In particular, $G_{i,j}$ are often assumed to be "shift" matrices. The linear deterministic model captures wireless signal interaction like interference and broadcast and on the other hand has an algebraic structure that can be exploited for understanding schemes in this network.

Corollary 2. *For the linear deterministic broadcast network the capacity region is in fact the cut-set bound.*

Proof. We can show that for the linear deterministic network the cut-set bound, $\bar{C}_{\mathcal{J}}(Q)$, is maximized by uniform and independent distribution of $\{X_v | v \in \mathcal{V}\}$. Therefore

$$\bar{C}_{\mathcal{J}}(Q) = \bar{C}_{\mathcal{J}}(Q_p) = \min_{\Omega \in \Lambda_{\mathcal{J}}} \operatorname{rank}(G_{\Omega,\Omega^c}), \tag{38}$$

where G_{Ω,Ω^c} is the matrix relating the vector of all the inputs at the nodes in Ω to the vector of all the outputs in Ω^c induced by (37). The inner bound has already been shown for the general deterministic network in Theorem 2. This proves the corollary. □

An important question is whether simpler linear schemes are optimal for these networks. It has already been shown in [1] that for the single-source single-destination relay network, linear mappings at all nodes suffice. The intuition behind the proof is that, when the relay nodes randomly pick transformation matrices, the resulting matrix between the source and the destination has rank equal to the min-cut rank of the network, with high probability. Therefore, if the rate is lesser than the min-cut rank, random linear coding at all nodes (including the source but not the destination) ensures an end-to-end full-rank matrix and the destination, knowing all these encoding matrices, picks up a decoding matrix, which is the inverse of the end-to-end matrix. This intuition is then used to obtain schemes in the general deterministic relay network and the Gaussian relay network in [1], where the relays perform random mapping operations resulting in an induced end-to-end channel between the source and the destination. Then the source uses a random code to map the messages, and the destination performs a typical set decoding. It has also been shown in [13] and [14], that for the linear deterministic relay network, restricting the relay mappings to permutation matrices is without loss of optimality. The next corollary claims a similar result even for the linear deterministic broadcast network.

Corollary 3. *For linear deterministic broadcast network, linear coding at every node is sufficient to achieve capacity. Furthermore, the mapping at relay nodes can be restricted to permutation matrices.*

Although this can be proved directly, we will use the connection between linear coding and reciprocity to prove this in the next section.

4.2 Reciprocity

The reciprocal of a Gaussian communication network (with unit power constraint at all nodes) with multiple unicast flows can be defined as the network where the roles of the sources and the destinations are swapped. Note that any channel coefficient that captures the signal attenuation between a pair of nodes is the same in either direction. For a linear deterministic network, the reciprocal network was defined in [15] as the network where the roles of the sources and the destinations are swapped, and the channel matrices are chosen as transposes of each other in the forward channel for the network and its reciprocal.

While it is unresolved whether a given network and its reciprocal have the same capacity region, many interesting examples are known for which this is true. For some cases, this reciprocity is applicable even at the scheme level.

- *Wire-line networks* can be considered as a special case of wireless networks studied here. It has been shown in [19] that wire-line networks are reciprocal (also called reversible in the literature) under linear coding.
- In [15], it was shown that reciprocity, under linear coding, can be extended naturally to the *linear deterministic network*. The reciprocity was shown at the scheme level and the coding matrices at each node can be obtained from the reciprocal network.
- In Gaussian networks, duality has been shown, [20, 21], between the multiple access channel (MAC) and broadcast channel (BC), where it was shown that the capacity region of the MAC is equal to the capacity region of the BC under the same sum power constraint. This duality was also shown, interestingly, at the scheme level between the dirty-paper pre-coding for the BC and the successive cancellation for the MAC.

The reciprocal network corresponding to the broadcast network studied here is the network with many sources and one destination.

4.2.1 Sufficiency of Linear Coding in LDN

The multi-source single-destination network has been studied in [6, 16, 17, 22], the capacity region for the linear deterministic network with many sources and one destination is established and it is further shown that linear coding is sufficient to achieve this. This is done by converting the problem to the case of single-source single-destination by adding a super-node and connecting all the source nodes to the super-node by orthogonal links with capacities equal to the rate required for that source. Since random linear coding at the source and the relays works for the single-source single-destination network, it works for this network too. Therefore, the source nodes and the relay nodes perform random mappings, and the destination, knowing the source and relay mappings, can then carefully pick the decoding matrix that inverts this overall matrix. Since this coding is linear, we can use the reciprocity result of [15], to show that any rate achievable in the dual multiple-source single-destination network is also achievable in the single-source multiple-destination case. Along with the fact that the cuts are reciprocal in these two networks, this implies that linear coding is optimal even in the case of the linear deterministic broadcast network. This result has also been shown in [17] without using reciprocity by adopting an algebraic approach.

Furthermore, from the results of [13] and [14], it can also be shown that the sources and the relays can pick up specific permutation matrices for the single-source single-destination network. The above argument can then the extended to show that a coding scheme involving only permutation mappings at the relays is sufficient for the linear deterministic broadcast network.

4.3 Receiver-Centric Vs. Transmitter-Centric Schemes: Intuition for the Gaussian Broadcast Network Scheme

We now continue on our discussion on duality for linear deterministic network to illustrate how these ideas lead us to a scheme for the Gaussian broadcast network. We begin by defining two viewpoints in which schemes can be constructed. A *transmitter-centric scheme* in one in which the scheme is constructed from the viewpoint of the transmitter, where the codebook at the transmitter is first selected using a random coding argument and then the receiver chooses its de-codebook in accordance with the realization of the transmit codebook. In contrast, in a *receiver-centric scheme*, we fix the decode-book, which comprises of the mappings from the received vectors to the messages, and based on these mappings, the transmitter chooses its codebook to ensure low probability of error.

Because random coding is done at the source, we can think of this scheme as first constructing the transmitter codebook in a random manner and the receiver then constructs its de-codebook as a function of the realization of the transmit codebook. While the scheme for the linear deterministic multi-source network is transmitter-centric, the scheme for the linear-deterministic broadcast network is *receiver-centric*.

4.3.1 A Point-to-Point Channel

For a point-to-point channel, the usual random coding scheme [4] can be regarded as either a transmitter-centric scheme, which is the traditional viewpoint (since the random codebook is thought of as being constructed at the source), or as a receive-centric scheme. It can be viewed as a receiver-centric scheme, because, at the receiver we construct a vector quantization codebook (alternately viewed as the decode-book) or rate R, which "quantizes" the received signal y^T to a vector $x^T(m)$, for some m where m is the message index and $x^T(m)$ is the m-th quantization codeword. Now the source sets its codebook to be equal to the vector quantization codebook at the destination. This scheme is the same scheme as the usual random coding scheme. The distinction between transmitter-centric and receiver-centric schemes in this example is therefore one of personal preference, rather than an enforced one.

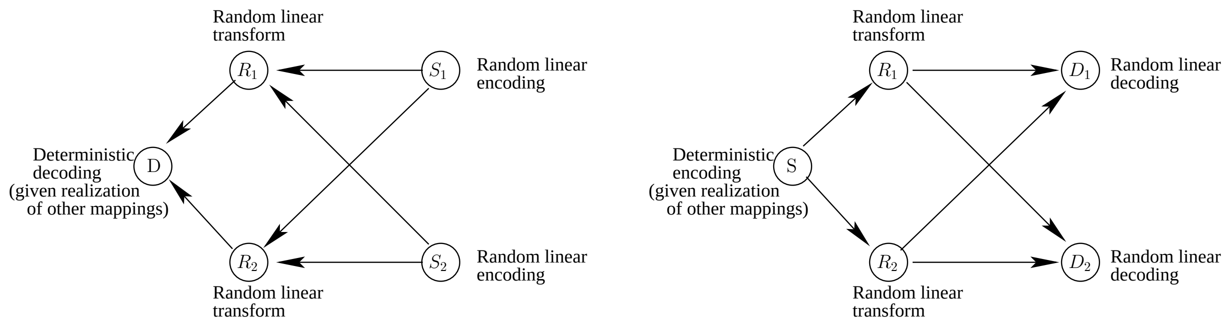
4.3.2 Multiple-Access Vs. Broadcast Channel

In some networks, we may not have the luxury to use the two viewpoints simultaneously, in which case we need to choose between the two. In the capacity-achieving coding scheme for the multiple access channel [4], the random coding is done at the transmitters and the receiver does joint typical-set decoding, based on the specific codebooks constructed at the sources. This provides a good example of a transmitter-centric scheme.

In contrast, for the two-user broadcast channel, we can now view the Marton coding scheme ([7, 8]), used in Sec. 2.2.2, as a receiver-centric scheme. In this scheme, there are two auxiliary random variables, U_1 and U_2, which we view as corresponding to the vector quantization variables at the two users. The receiver i can be thought of as constructing a vector quantization codebook which "quantizes" the received vector Y_i^T to $U_i^T(w_i)$, where w_i is an index belonging to a set larger than the set of all messages to user i, and bins the set of all w_i to the message m_i for user i. The transmitter, to transmit a message pair (m_1, m_2), finds a pair (w_1, w_2) such that $U_1^T(w_1), U_2^T(w_2)$ is jointly typical. From this viewpoint, the receivers are choosing random de-codebooks, and the transmitters are choosing specific codebooks to be a function of the realization of the de-codebook. Thus the coding scheme can be viewed as a receiver-centric one.

4.3.3 Multiple-Access Vs. Broadcast in Linear-Deterministic Networks

From [16, 6], we know that a transmitter-centric scheme, where the sources and the intermediate nodes perform random coding, is optimal for the many-source single-destination problem in the linear determinstic setup. Intuition suggests that a natural receiver-centric method should work for the reciprocal network (i.e., single-source multiple-destination network). In particular, the relays perform random mappings, and the destinations perform "random decoding", i.e., they fix a random linear mapping from the received vector into a smaller message vector. Once these mappings at the relay and the destinations are fixed, the source evaluates the induced linear broadcast channel between the source and the various destinations; and constructs a linear broadcast code for this channel. This scheme can then be shown to be



(a) Random coding scheme for multi-source single-destination network

(b) Dual random coding scheme for single-source multi-destination network

Figure 5: Reciprocity in linear deterministic networks

optimal for the broadcast network, because this is the reciprocal of the linear random coding scheme, which is optimal for the multi-source single-destination network, as shown in Fig. 5.

4.3.4 Scheme for Gaussian Broadcast Networks: Lifting Scheme as a Receiver-Centric Scheme

The general idea for the scheme for the linear deterministic broadcast network is the foundation of our scheme for the Gaussian broadcast network in Sec. 3. In order to build the scheme for the Gaussian network, we first construct a scheme for general deterministic networks (of which the DSN is a special case) and then lift the scheme from the DSN to the Gaussian network. In case of the linear-deterministic broadcast network, the source-mapping depended on the specific relay transformations used, not just on the probability distribution used to create the relay transformation. Extending this idea, we would like to construct a scheme for the general deterministic network, where the source codebook is a function of the *specific relay transformation*. Indeed, we resolve this problem by constructing a Marton scheme at the source for the vector broadcast channel induced by the specific relay mappings.

Next, the scheme for lifting codes from the DSN to Gaussian relay networks proposed in [3] requires each node, including the destination, to prune their received vectors to a restricted set to ensure that the received vector in the DSN can be decoded from the received vector in the Gaussian network. Since this scheme restricts the received codewords at the destination, this scheme also naturally fits into a receiver-centric viewpoint.

While the lifting procedure proposed there works only for single-source single-destination networks, we extend the procedure to our specific scheme for broadcast network. We achieve this by designing a pruned Marton code, in which the receivers are guaranteed to receive vectors which are in the pruned set. Instead of binning the set of all possible received vectors into messages, as we would for a broadcast channel, we now bin only the pruned received vectors to construct the pruned Marton coding scheme. The natural alignment of the receiver-centric viewpoints of the Marton scheme and the lifting scheme allows us to construct the scheme for the Gaussian broadcast network.

4.4 Approximate Reciprocity in Gaussian Multi-Source and Broadcast Networks

In this section, we will demonstrate that there is an approximate reciprocity in the capacity regions of a Gaussian multi-source network and the corresponding reciprocal Gaussian broadcast network.

In our model, we have assumed, without loss of generality, the average transmit power constraint of unity at each node. We have also assumed that the reciprocal network, in addition to having the same channel coefficients, also has unit power constraints at each node. However, it is not clear if this is the "right" way of defining the corresponding reciprocal network. For instance, in [20, 21], MAC-BC duality was shown under the assumption of same total transmit power in both networks; however this power could be divided amongst the nodes in a different manner in the forward and reciprocal networks. Under this assumption, it was shown that the capacity region of the two networks was identical. However, since we are concerned only about approximate reciprocity in this section, which is a weaker form of reciprocity, our definition of unit power constraint everywhere will be sufficient to show approximate reciprocity.

In [16] and [6], a coding scheme is given for the Gaussian network with many sources and is shown to achieve the cut-set bound region within a constant gap, which depends only on the network gain. In Sec. 3, we have showed that for the Gaussian broadcast network also, we can achieve the cut-set bound region within a constant gap. As a result, to show that the capacity region of the two networks are themselves within a constant gap, which depends only on the network topology and not on the channel gains, all we need to do is to observe that cut-sets of the reciprocal networks are within a constant gap of each other. Note that the cut-set bounds corresponds to MIMO point-to-point channel where all the nodes on the source side of the nodes can be thought of as transmit antennas and all the nodes on the destination side can be thought of as receive antennas. The relationship then between a cut in a network and the corresponding cut in the reciprocal network is the same as the relationship between a MIMO channel with channel matrix H and the reciprocal MIMO channel with the channel matrix H^T. The reciprocity of MIMO channel has been shown in [23], under equal total transmit power, i.e. the capacity of the two networks is the same. It can be further shown that restricting to per node power constraint only leads to a loss which does not depend on the channel gains. Therefore, we can show that the cut-set bounds are reciprocal.

4.5 Induced Coordination in Relays' Transmission

Let us consider a simple example for a broadcast relay network comprised of a single source, two relays and two destinations, shown in Fig. 6. The link between the source to the two relays is infinite, which implies therefore that the network is essentially a MISO broadcast channel with two transmit antennas and two receivers, each with a single antenna. It is clear that for a MISO broadcast channel, independent coding across the two relays is insufficient to even obtain the best possible degrees-of-freedom. Therefore, any scheme that is approximately optimal needs to perform coordinated transmission at the relays.

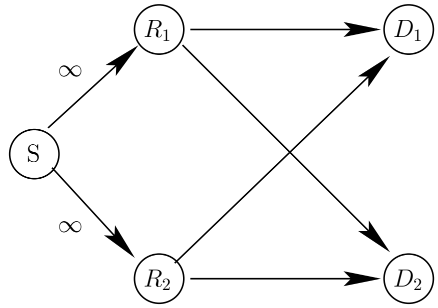


Figure 6: MISO broadcast channel as a special case of broadcast network

In the proposed scheme, the relays perform quantization followed by independent encoding of the quantized bits into transmitted vectors. At a first glance, a scheme in which the relays are performing independent mappings seems incapable of attaining good performance because of the inability to induce coordination. However two key features in the proposed scheme help avoid this pitfall.

- The relays R_1 and R_2 perform quantize-and-encode relaying in the aforementioned example, in spite of the fact that they can decode the source message completely. Had the relays decoded the source message and performed independent encoding, there is no possibility of achieving the degrees-of-freedom of even this simple broadcast network.
- The source takes into account the specific realizations of the relay mappings and constructs the coding scheme. This ensures that from the point of view of the receiver, the signals transmitted by the two relays appear coordinated. In particular, in this example, since the channel from S to R_i is infinitely good, the relay R_i quantizes the received signal to a very fine degree and encodes this for transmission to the destination. This gives the source many degrees-of-freedom to encode information in the various least-significant-bits of its transmission, so that after the relay mappings, the relay transmissions appear coordinated.

4.6 Channel State Information

We now examine the channel state information required at the various nodes for the schemes proposed in Sec. 2 and Sec. 3 for deterministic and gaussian broadcast networks.

4.6.1 Deterministic Broadcast Networks

For deterministic networks, the following channel state information is required:

1. All non-source non-destination nodes are unaware of any channel state information.
2. We assume that each destination knows the distribution of the received vector $\vec{Y}_{D_i}$, and the transmitted rate $R_i(t_3)$ for each t_3. The destination bins the set of all typical vectors into $2^{T_1 T_2 R_i}$ bins corresponding to the messages, and uses this as the decoding rule.

3. The transmitter is assumed to have full CSI, and knows the relay mappings at all nodes and also the binning scheme at the destinations. The transmitter construct the codebook using the same binning scheme as the receiver.

Thus this scheme has the interesting property that if the transmitter had all knowledge, the intermediate nodes have zero knowledge and the destination has a little knowledge (about the distribution of the received vector), then the same rate can be achieved as the complete CSI case.

This is dual to the situation of the multi-source single-destination network, where the receiver having full knowledge, intermediate nodes having zero knowledge and the transmitters having a little knowledge (about the distribution of the transmitted vector) can achieve the same rate as full channel knowledge.

4.6.2 Gaussian Broadcast Network

For Gaussian networks, the following channel state information is required:

1. All non-source non-destination nodes are unaware of any channel state information. Node v however knows the probability distribution of the *received and transmitted vectors* p_{Y_v}, p_{X_v} for the corresponding DSN, which will be used to calculate the relay mappings. The node v also needs to use the received vector distribution to pick a pruned subset of the typically received vectors in the corresponding DSN.

2. We assume that each destination knows the distribution of the received vector in the corresponding DSN $\vec{Y}_{D_i}$, and the transmitted rate $R_i(t_3)$ for each t_3. The destination maps bins the set of all typical vectors into $2^{T_1 T_2 R_i}$ bins corresponding to the messages, and uses this as the decoding rule.

3. The transmitter is assumed to have full CSI, and knows the mappings used at all the nodes and also the binning scheme at the destinations. The transmitter then uses the same binning scheme used at the receiver.

This scheme has the interesting property that if the transmitter had all knowledge, the intermediate nodes and the destination have some knowledge, then the same rate can be achieved as the complete CSI case.

5 Generalizations

In this section, we present various generalizations of our result, for half-duplex networks in Section 5.1, for networks with multiple antenna in Section 5.2 and for broadcast wireless networks, where some set of nodes demand the same information and other nodes demand independent information in Section 5.3.

5.1 Half Duplex Networks

Our discussion so far has been restricted to the context of full duplex scenario. A network is said to be *half duplex* if the nodes in the network can either transmit or receive information, but not do both simultaneously. Therefore the network needs to be *scheduled* by specifying which nodes are listening and which nodes are transmitting at any given time instant. Let the set of all possible half-duplex schedules at any time instant be $\mathcal{H}$. An edge e_{ij} is said to be *active* at time slot k if v_i is transmitting and v_j is receiving at that time slot.

Consider K time slots and at any time instant k, let $h_k \in \mathcal{H}$ be the half duplex schedule used, and h^K be the sequence $h_1, h_2, ..., h_K$. We consider only *static schedules* here, that is, schedules that are specified apriori and do not vary depending on dynamic parameters like channel noise. For any static schedule h^K, we can unfold the network graph with respect to that schedule. This procedure is performed in [1], and is the same as the procedure in Section 3.2, except for the following difference: $v_1[k]$ is connected to $v_2[k+1]$ with a link h_{v_1,v_2} only when $e_{v_1 v_2}$ is active at time slot k.

Given that the network is operated under a schedule h^K, we define the set of all rate pairs achievable as the *capacity region under the schedule* h^K. An upper bound on the capacity region under the schedule h^K is given by the cut-set bound in the unfolded layered network corresponding to the schedule. This rate can be achieved within a constant gap by using Theorem 1. Thus for any schedule h^K, any rate tuple within the constant $k = O(|\mathcal{V}| \log(|\mathcal{V}|))$ of the cut-set bound can be achieved (to within a constant number of bits) using that schedule and then using the scheme of Theorem 1 for the unfolded layered network. Now, we can optimize over all schedules $h^K \in \mathcal{H}$ allowed under the half-duplex constraints. Thus, the capacity region of the network under static half-duplex scheduling is the union over all possible schedules of the capacity region under schedule h^K. Therefore, any rate tuple $(R_1, ..., R_J)$ such that $(R_1+k, R_2+k, ..., R_J+k)$ is in the capacity region of the network under static half-duplex scheduling can be achieved by using the method described here.

5.2 MIMO

In this section, we consider the implication of having multiple antenna elements at each of the nodes in the network. Suppose v possesses m_v antenna elements, which are used for both transmission and reception. The basic result for multi-antenna broadcast networks is the following.

Theorem 3. *For the multi-antenna broadcast network, a rate vector* $(R_1, \ldots, R_J)$ *is achievable,*

$$(R_1 + k, \ldots, R_J + k) \in \bar{\mathcal{C}} \tag{39}$$

for some constant k*, which depends only on the number of nodes, and not on the channel coefficients, and* $k = O(M \log M)$ *where* $M = \sum_{v \in \mathcal{V}} m_v$ *.*

Proof. The proof is essentially the same as the one for the single antenna case in Section 3. We only outline the proof below, highlighting the key distinctions.

1. For a given multi-antenna Gaussian network, we first obtain a multi-antenna DSN such that the cut-set bound for the Gaussian network and the cut-set bound under the product form distribution for the DSN differ only by a constant (in the same manner as we obtained Lemma 2 in Sec. 3.1.1. This is because (72) continues to hold for multi-antenna networks, with $|\mathcal{V}|$ now replaced by the total number of antennas M.

2. Then we use Theorem 2 to show that the cut-set bound under product form distribution is achievable for the DSN.

3. We can now prune the codebook for the DSN by a factor κ, to get a pruned scheme for the DSN, $\mathcal{P}_\kappa$. The pruned scheme achieves a rate $M\kappa$ lesser than the original rate.

4. Under the pruned scheme, the received vector in the DSN can be decoded from the received vector in the Gaussian network.

5. Therefore the DSN coding scheme can be emulated in the Gaussian network, and achieves a rate a constant $k = O(M \log M)$ lesser than the cut-set bound in the Gaussian network.

□

5.3 Broadcast-cum-Multicast

The broadcast network comprised of a single source S and destinations $D_1, D_2, ..., D_J$ demanding independent messages at rates $R_1, R_2, ..., R_J$. Suppose that in addition there are also other multicast destinations $M_1, M_2, ..., M_L$ that demand all the messages transmitted by the source. We call such a network a broadcast-cum-multicast network. In this section, we will show that even for such networks, the cut-set bound is achievable to within a constant number of bits. This network is a generalization of both the multicast network considered in [1] and the broadcast network considered in the previous sections.

First we note that the cut-set bound for the broadcast-cum-multicast network is given by the cut-set bound for the broadcast network, along with the cut-set constraints for each multicast receiver. In particular for the Gaussian broadcast-cum-multicast network, if $(R_1, ..., R_J)$ is achievable, then $\forall\ \mathcal{J} \subseteq [J]$ there exists a joint distribution Q such that,

$$R_{\mathcal{J}} \leq \min_{\Omega \in \Lambda_{\mathcal{J}}} I(X_\Omega; Y_{\Omega^c} | X_{\Omega^c}), \tag{40}$$

and in addition, the sum rate is constrained by all the multicast destinations since all these destinations demand all the messages transmitted by the source

$$R_{[J]} \leq \min_{i \in [L]} \min_{\Omega \in \Lambda_{M_i}} I(X_\Omega; Y_{\Omega^c} | X_{\Omega^c}). \tag{41}$$

The set of all rate tuples inside the cut-set bound is then denoted by $\bar{C}$.

The main result for the wireless broadcast-cum-multicast network is that any rate a constant away from cut-set bound is achievable.

Theorem 4. *For the Gaussian broadcast-cum-multicast network, there exists a constant k, which does not depend on the channel coefficients and is $O(|\mathcal{V}|\log|\mathcal{V}|)$, such that a rate vector of $(R_1, ..., R_L)$ is achievable whenever*

$$(R_1 + k, ..., R_J + k) \in \bar{C}. \tag{42}$$

To prove this result, we follow an approach similar to the one we took for broadcast networks. First we will prove a result for deterministic broadcast-cum-multicast networks. Second, we show that the Gaussian network can emulate the deterministic superposition network with a constant rate loss. These two steps are completed in the rest of this section.

5.3.1 Deterministic Broadcast-cum-Multicast Network

The next Lemma shows that for the deterministic broadcast-cum-multicast network, the cut-set bound evaluated under product form distributions is achievable.

Lemma 9. *For the deterministic broadcast-cum-multicast network, the cut-set bound under product-form distributions is achievable, i.e., a rate vector $(R_1, ..., R_K)$ is achievable if for every $\mathcal{J} \in [J]$ there is some product probability distribution Q_p, such that,*

$$\begin{aligned} R_{\mathcal{J}} &\leq \min_{\Omega \in \Lambda_{\mathcal{J}}} I\left(X_\Omega; Y_{\Omega^c} | X_{\Omega^c}\right) && (43)\\ &= \min_{\Omega \in \Lambda_{\mathcal{J}}} H\left(Y_{\Omega^c} | X_{\Omega^c}\right), \quad \text{and} && (44)\\ R_{[J]} &\leq \min_{i \in [L]} \min_{\Omega \in \Lambda_{M_i}} I\left(X_\Omega; Y_{\Omega^c} | X_{\Omega^c}\right) && (45)\\ &= \min_{i \in [L]} \min_{\Omega \in \Lambda_{M_i}} H\left(Y_{\Omega^c} | X_{\Omega^c}\right). && (46) \end{aligned}$$

Proof. Coding Scheme: The source operation, relaying operations and decoding operations at the broadcast destinations remain the same as in Section 2.2. In addition, we need to specify the decoding operation at the multicast destinations.

At each level-3 block, each multicast destination performs a typical set decoding with the set of all possible $\vec{X}_s^{T_2}$ (corresponding to all possible messages) and finds the unique $W_1(t_3), ..., W_J(t_3)$ for which the $(\vec{X}_s^{T_2}(W_1(t_3), ..., W_J(t_3)), \vec{Y}_{M_i}^{T_2}) \in T_\delta^{T_2}(\vec{X}_s, \vec{Y}_{M_i})$.

Performance Analysis: If the cut-set bound for the broadcast destinations is satisfied, then the probability of error at these destinations is guaranteed to be small (by the same analysis in Sec 2.3). We need to analyze the error events at all multicast destinations. The probability of error at destination M_i in the t_3-th level-2 block goes to zero, with $T_2 \to \infty$ if

$$\begin{aligned} R_{[J]}(t_3) &< \frac{1}{T_1} I(\vec{X}_s; \vec{Y}_{M_i} | F_r = f_r(t_3)) && (47)\\ &= \frac{1}{T_1} H(\vec{Y}_{M_i} | F_r = f_r(t_3)). && (48) \end{aligned}$$

As before, since the overall rate is given by averaging the rate across all T_3 blocks, as $T_3 \to \infty$, the probability of error at destination M_i is small if,

$$R_{[J]} < \frac{1}{T_1} H(\vec{Y}_{M_i} | F_r). \tag{49}$$

Using Lemma 1, we can relate the entropy term above to the cut-set w.r.t. this destination, i.e for any arbitrary $\epsilon > 0$, $\exists T_1$, s.t., we have

$$\frac{1}{T_1} H(\vec{Y}_{M_i}|F_r) \geq \min_{\Omega \in \Lambda_{M_i}} H\left(Y_{\Omega^c}|X_{\Omega^c}\right) - \epsilon. \tag{50}$$

□

Corollary 4. *For the linear deterministic broadcast-cum-multicast network, the cut-set bound is achieved. For the deterministic broadcast-cum-multicast channel (a deterministic broadcast-cum-multicast network in the absence of relays and destination cooperation) the cut-set bound is achieved.*

Proof. In the former case, the cut-set bound under product form distribution is the same as the cut-set bound under general distributions since there is only one transmitting node in the network. The latter case can be proved by showing the cut-set bound for linear deterministic networks is optimized by product form distributions. □

5.3.2 Gaussian Broadcast-cum-Multicast Network

We will now prune and lift the coding scheme from the DSN broadcast-cum-multicast network to the Gaussian broadcast-cum-multicast network. Since the encoding at the source and relay mappings are the same as a network with the broadcast destinations alone, the same procedure used for pruning and lifting the broadcast code in Sec. 3.1.2 and Sec. 3.1.3 can be used to lift the code for the DSN broadcast-cum-multicast network to the Gaussian broadcast-cum-multicast network. This procedure ensures that all nodes in the Gaussian network can decode the corresponding received vector in the DSN. Therefore, the destinations carry out the same decoding operation that they perform in the DSN. The performance analysis of the scheme is similar to the performance analysis for the broadcast network in Sec. 3, and it can be shown that any rate tuple $(R_1, .., R_J)$ that satisfies $(R_1 + k, ..., R_J + k) \in \bar{C}$ can be achieved, where $k = O(|\mathcal{V}| \log(|\mathcal{V}|))$.

A Proof of Lemma 1

Proof. Fix any $\mathcal{J} \subseteq \{1, \ldots, J\}$. To prove the lemma, we consider a communication scenario where the source needs to send a message to a single destination which has access to $Y_{D_{\mathcal{J}}}$ with rate $\tilde{R}_{(\mathcal{J})}$. Define,

$$\tilde{W}_{(\mathcal{J})} \sim \text{ Uniform } \left\{[2^{\tilde{R}_{(\mathcal{J})}}]\right\}, \tag{51}$$

and the mapping at the source node,

$$F_{(\mathcal{J})} : \tilde{W}_{(\mathcal{J})} \to \mathcal{X}_S^T, \text{which is generated using i.i.d. } p(X_S). \tag{52}$$

Note that $F_{(\mathcal{J})}$ denotes a random source code-book and $F_{\mathcal{V}_R}$ denotes random relay mappings. The probability of error conditioned on a given source code-book and relay mapping is given by,

$$\mathbb{P}\left\{\mathcal{E}|F_{(\mathcal{J})},F_{\mathcal{V}_R}\right\}=\mathbb{P}\left\{Y^T_{D_\mathcal{J}}(\tilde{W}_\mathcal{J})=Y^T_{D_\mathcal{J}}(w')|w'\neq\tilde{W}_\mathcal{J},F_{(\mathcal{J})},F_{\mathcal{V}_R}\right\}, \tag{53}$$

and the average probability of error, averaged across all code-books and random-relay mappings, is given by,

$$\mathcal{P}_e\stackrel{\text{def}}{=}\mathbb{E}\left[\mathbb{P}\left\{\mathcal{E}|F_{(\mathcal{J})},F_{\mathcal{V}_R}\right\}\right]. \tag{54}$$

By the coding theorem in [1], we have

$$\mathcal{P}_e\to 0,\ \text{as } T\to\infty,\ \forall\ \tilde{R}_{(\mathcal{J})}<\bar{C}_\mathcal{J}(Q_p). \tag{55}$$

Now,

$$\begin{aligned}
I(\tilde{W}_{(\mathcal{J})};Y^T_{D_\mathcal{J}}|F_{(\mathcal{J})},F_{\mathcal{V}_R}) &= H(\tilde{W}_{(\mathcal{J})})-H(\tilde{W}_{(\mathcal{J})}|Y^T_{D_\mathcal{J}},F_{(\mathcal{J})},F_{\mathcal{V}_R})\\
&= T\tilde{R}_{(\mathcal{J})}-\mathbb{E}\left[H(\tilde{W}_\mathcal{J})-H(\tilde{W}_{(\mathcal{J})}|Y^T_{D_\mathcal{J}},F_{(\mathcal{J})}=f_{(\mathcal{J})},F_{\mathcal{V}_R}=f_{\mathcal{V}_R})\right]\\
&\stackrel{\text{Fano}}{\geq} T\tilde{R}_{(\mathcal{J})}-\mathbb{E}\left[1+\mathbb{P}\left\{\mathcal{E}|F_{(\mathcal{J})},F_{\mathcal{V}_R}\right\}\tilde{R}_{(\mathcal{J})}T\right]\\
&= T\tilde{R}_{(\mathcal{J})}-(1+\mathcal{P}_e\tilde{R}_{(\mathcal{J})}T).
\end{aligned}$$

Letting $\tilde{R}_{(\mathcal{J})}=\bar{C}_\mathcal{J}(Q_p)-\epsilon_1$ and for large enough T, we have

$$I(\tilde{W}_{(\mathcal{J})};Y^T_{D_\mathcal{J}}|F_{(\mathcal{J})},F_{\mathcal{V}_R})\geq T(\bar{C}_\mathcal{J}(Q_p)-\epsilon). \tag{56}$$

Further

$$\begin{aligned}
I(\tilde{W}_{(\mathcal{J})};Y^T_{D_\mathcal{J}}|F_{(\mathcal{J})},F_{\mathcal{V}_R}) &= H(Y^T_{D_\mathcal{J}}|F_{(\mathcal{J})},F_{\mathcal{V}_R})-H(Y^T_{D_\mathcal{J}}|F_{(\mathcal{J})},F_{\mathcal{V}_R},\tilde{W}_\mathcal{J})\\
&\leq H(Y^T_{D_\mathcal{J}}|F_{\mathcal{V}_R})-H(Y^T_{D_\mathcal{J}}|F_{(\mathcal{J})},F_{\mathcal{V}_R},\tilde{W}_\mathcal{J},X^T_S)\\
&= I(X^T_S;Y^T_{D_\mathcal{J}}|F_{\mathcal{V}_R}).
\end{aligned}$$

Therefore,

$$I(X^T_S;Y^T_{D_\mathcal{J}}|F_{\mathcal{V}_R})\geq T(\bar{C}_\mathcal{J}(Q_p)-\epsilon). \tag{57}$$

Since the channel is deterministic

$$\begin{aligned}
H(Y^T_{D_\mathcal{J}}|F_{\mathcal{V}_R}) &= I(X^T_S;Y^T_{D_\mathcal{J}}|F_{\mathcal{V}_R})\\
&\geq T(\bar{C}_\mathcal{J}(Q_p)-\epsilon).
\end{aligned}$$

The proof is completed by choosing T_1 to be the maximum T over all $\mathcal{J}$. □

B Proof of Lemma 3

Proof. Throughout this section, we will assume that the relay mappings are fixed to $f_{\mathcal{V}_\mathcal{R}}(t_3)$ without making this explicit in the conditioning expressions. We will assume $N=2$ to begin with and establish a lower bound on $|\mathfrak{Z}_1|$ (w.l.o.g.).

$$\mathfrak{Z}_1 = \{\vec{y}_1^{T_2}\in\mathfrak{S}_1:\exists\vec{y}_2^{T_2}\in\mathfrak{S}_2,(\vec{y}_1^{T_2},\vec{y}_2^{T_2})\in\mathcal{T}_\delta^{T_2}\}.$$

We will consider two cases.

Case 1: $H(\vec{Y}_2|\vec{Y}_1) > T_1\kappa$

Fix a $\vec{y}_1^{T_2} \in \mathfrak{S}_1$. The set of sequences $\vec{y}_2^{T_2}$ which are jointly typical with $\vec{y}_1^{T_2}$ is given by $\mathcal{T}_\delta^{T_2}(\vec{Y}_2|\vec{y}_1^{T_2})$, and its size is of the order $2^{T_2 H(\vec{Y}_2|Y_1)}$. Now $\mathfrak{S}_2$ is a random $2^{-T_2 T_1 \kappa}$ fraction of the typical set $\mathcal{T}_\delta^{T_2}(\vec{Y}_2)$. Therefore we can show that , if $H(\vec{Y}_2|\vec{Y}_1) > T_1\kappa$, then w.h.p. as $T_2 \to \infty$,

$$\mathfrak{S}_1 \cap \mathcal{T}_\delta^{T_2}(\vec{Y}_2|\vec{y}_1^{T_2}) \neq \emptyset, \tag{58}$$

i.e. every $\vec{y}_1^{T_2} \in \mathfrak{S}_1$ will be jointly typical with some $\vec{y}_2^{T_2} \in \mathfrak{S}_2$. Therefore the size of the set is $|\mathfrak{Z}_1| \doteq |\mathfrak{S}_1| \doteq 2^{T_2(H(\vec{Y}_1) - T_1\kappa)}$ w.h.p. as $T_2 \to \infty$. Clearly

$$|\mathfrak{Z}_1| \dot{>} 2^{T_2(H(\vec{Y}_1) - 2T_1\kappa)}. \tag{59}$$

Case 2: $H(\vec{Y}_2|\vec{Y}_1) \leq T_1\kappa$

Fix a $\vec{y}_1^{T_2} \in \mathfrak{S}_1$. For T_2 large enough, the probability that there exists a sequence $\vec{y}_2^{T_2} \in \mathfrak{S}_2$ which is conditionally typical given any $\vec{y}_1^{T_2} \in \mathfrak{S}_1$ is given by $p \doteq 2^{T_2(H(\vec{Y}_2|\vec{Y}_1) - T_1\kappa)}$. Consider an arbitrary subset $\mathfrak{S}_{11} \subseteq \mathfrak{S}_1$ of size $2^{T_2\Delta}$. The probability that there is an element in $\mathfrak{S}_{11}$ which is jointly typical with an element in $\mathfrak{S}_2$ is given by:

$$\begin{aligned}
\mathbb{P}\{\exists\, \vec{y}_1^{T_2} \in \mathfrak{S}_{11} : (\vec{y}_1^{T_2}, \vec{y}_2^{T_2}) &\in \mathcal{T}_\delta^{T_2}(\vec{Y}_1, \vec{Y}_2) \text{ for some } \vec{y}_2^{T_2} \in \mathfrak{S}_2\} \\
&= 1 - (1-p)^{2^{T_2\Delta}} && (60) \\
&\geq 1 - e^{-p 2^{T_2\Delta}} && (61) \\
&= 1 - e^{-2^{T_2(H(\vec{Y}_2|\vec{Y}_1) - T_1\kappa + \Delta)}} && (62) \\
&\to 1, \text{ if } \Delta > T_1\kappa - H(\vec{Y}_2|\vec{Y}_1), && (63)
\end{aligned}$$

as $T_2 \to \infty$.

So we will set $\Delta = T_1\kappa - H(\vec{Y}_2|\vec{Y}_1) + \epsilon_1$. Thus if we divide $\mathfrak{S}_1$ into disjoint sets $\mathfrak{S}_{1i}$ each of size $2^{T_2\Delta}$, then each will have at least one element in $\mathfrak{Z}_1$. Therefore

$$\begin{aligned}
|\mathfrak{Z}_1| &\geq \frac{|\mathfrak{S}_1|}{2^{T_2\Delta}} && (64) \\
&\doteq \frac{2^{T_2(H(\vec{Y}_1) - T_1\kappa)}}{2^{T_2\Delta}} && (65) \\
&= 2^{T_2(H(\vec{Y}_1) - T_1\kappa - T_1\kappa + H(\vec{Y}_2|\vec{Y}_1))}. && (66)
\end{aligned}$$

Therefore

$$|\mathfrak{Z}_1| \dot{>} 2^{T_2(H(\vec{Y}_1) - 2T_1\kappa)}. \tag{67}$$

This completes the proof for the case when $N = 2$. By iterating this calculation, we can show that for a general N, w.h.p. as $T_2 \to \infty$,

$$|\mathfrak{Z}_1| \;\dot{>}\; 2^{T_2(H(\vec{Y}_1) - NT_1\kappa)}. \tag{68}$$

□

C Proof of Lemma 4

Proof. Let us consider the case where there are only two receivers i.e. $J = 2$. The proof extends similarly for the general case. $\mathfrak{Z}_{D_i}$ is the set of all typically received codewords at the destination D_i. We know from Lemma 3 that $|\mathfrak{Z}_{D_i}| \dot{>} 2^{-T_2 T_1 \kappa N} 2^{T_2 H(\vec{Y}_{D_i} | F_{\mathcal{V}_\mathcal{R}} = f_{\mathcal{V}_\mathcal{R}}(t_3))}$ w.h.p. as $T_2 \to \infty$.

Since we are binning the set of all $\vec{y}_{D_i}^{T_2} \in \mathfrak{Z}_{D_i}$ into $2^{T_2 T_1 R_i}$ bins, to ensure each bin has at the least one codeword, we need for some $\epsilon > 0$,

$$2^{T_2 T_1 (R_i(t_3) + \epsilon)} \quad \leq \quad |\mathfrak{Z}_{D_i}|.$$

It is sufficient to have,

$$\begin{aligned} 2^{T_2 T_1 (R_i(t_3) + \epsilon)} &\leq 2^{-T_2 T_1 \kappa N} 2^{T_2 H(\vec{Y}_{D_i} | F_{\mathcal{V}_\mathcal{R}} = f_{\mathcal{V}_\mathcal{R}}(t_3))} \\ \Rightarrow \; R_i(t_3) &< \frac{1}{T_1} H(\vec{Y}_{D_i} | F_{\mathcal{V}_\mathcal{R}} = f_{\mathcal{V}_\mathcal{R}}(t_3)) - N\kappa. \end{aligned}$$

Now, there is no error if corresponding to each message pair $(W_1(t_3), W_2(t_3))$, the source can find a jointly typical $(\vec{y}_{D_1}^{T_2}, \vec{y}_{D_2}^{T_2})$ in the bin corresponding to $(W_1(t_3), W_2(t_3))$. This can be done w.h.p. as $T_2 \to \infty$, as long as there are at the least $2^{T_2 (I(\vec{Y}_{D_1}; \vec{Y}_{D_2} | F_{\mathcal{V}_\mathcal{R}} = f_{\mathcal{V}_\mathcal{R}}(t_3)) + \epsilon)}$ pairs of $(\vec{y}_{D_1}^{T_2}, \vec{y}_{D_2}^{T_2})$ in this bin. This condition translates to

$$\frac{|\mathfrak{Z}_{D_1}|}{2^{T_2 T_1 R_1(t_3)}} \frac{|\mathfrak{Z}_{D_2}|}{2^{T_2 T_1 R_2(t_3)}} \quad \geq \quad 2^{T_2 (I(\vec{Y}_{D_1}; \vec{Y}_{D_2} | F_{\mathcal{V}_\mathcal{R}} = f_{\mathcal{V}_\mathcal{R}}(t_3)) + \epsilon)}.$$

This is satisfied if,

$$\begin{aligned} \frac{2^{T_2 \left\{ H(\vec{Y}_{D_1} | F_{\mathcal{V}_\mathcal{R}} = f_{\mathcal{V}_\mathcal{R}}(t_3))) + H(\vec{Y}_{D_2} | F_{\mathcal{V}_\mathcal{R}} = f_{\mathcal{V}_\mathcal{R}}(t_3)) \right\}}}{2^{T_2 T_1 (R_1(t_3) + R_2(t_3))} 2^{2 T_2 T_1 N \kappa}} &\geq 2^{T_2 (I(\vec{Y}_{D_1}; \vec{Y}_{D_2} | F_{\mathcal{V}_\mathcal{R}} = f_{\mathcal{V}_\mathcal{R}}(t_3)) + \epsilon)} \\ \Rightarrow R_1(t_3) + R_2(t_3) &< \frac{1}{T_1} H(\vec{Y}_{D_1}, \vec{Y}_{D_2} | F_{\mathcal{V}_\mathcal{R}} = f_{\mathcal{V}_\mathcal{R}}(t_3)) - 2 T_2 T_1 N \kappa. \end{aligned}$$

□

D Proof of Lemma 2

First of all, we note that the for the Gaussian network, the cut-set bound is given by letting the inputs be jointly Gaussian, i.e. $Q = \mathcal{CN}(0, K)$ with $K_{jj} \leq 1$ such that

$$\bar{\mathcal{C}}^{\text{Gauss}} \stackrel{\text{def}}{=} \left\{ (R_1, ..., R_J) : R_\mathcal{J} \leq \bar{C}_\mathcal{J}^{\text{Gauss}}(Q) = \min_{\Omega \in \Lambda_\mathcal{J}} \log |I + H_{\Omega, \Omega^c} K_\Omega H_{\Omega, \Omega^c}^*|, \; \forall \mathcal{J} \subseteq [J] \right\} \tag{69}$$

where H_{Ω, Ω^c} is defined as the matrix such that

$$Y_{\Omega^c} = H_{\Omega, \Omega^c} X_\Omega + H_{\Omega^c, \Omega^c} X_{\Omega^c} + Z_{\Omega^c}, \tag{70}$$

and K_Ω is the conditional covariance matrix of X_Ω given X_{Ω^c}.

It is also well-known (see Lemma 6.6 in [1]) that restricting to the product distribution $Q_p = \mathcal{CN}(0, I)$, only leads to relaxation of the mutual information term in (69) by at most $\min(|\Omega|, |\Omega^c|)$, and therefore

$$\begin{aligned} \bar{C}_{\mathcal{J}}^{\text{Gauss}}(Q) &\leq \bar{C}_{\mathcal{J}}^{\text{Gauss}}(Q_p) + |\mathcal{V}|/2 \\ &= \min_{\Omega \in \Lambda_{\mathcal{J}}} \log |I + H_{\Omega,\Omega^c} H^*_{\Omega,\Omega^c}| + |\mathcal{V}|/2. \end{aligned} \tag{71}$$

The following proposition shows that every cut in the Gaussian network is within a a constant gap to the cut in the DSN.

Proposition 1. *For every $\Omega \in \Lambda_{\mathcal{J}}$, there exists a product distribution $Q_p^{DSN} = \prod_{v \in \mathcal{V}} p(X_v^{DSN})$ such that,*

$$\log |I + H_{\Omega,\Omega^c} H^*_{\Omega,\Omega^c}| \leq I\left(Y_{\Omega^c}^{DSN}; X_\Omega^{DSN} | X_{\Omega^c}^{DSN}\right) + O(|\mathcal{V}|), \tag{72}$$

Proof. We will reduce the jointly Gaussian vector X to a vector that is valid in the DSN model and show that the reduction only leads to a $O(|\mathcal{V}|)$ loss in mutual information.

1. Note that

$$\log |I + H_{\Omega,\Omega^c} H^*_{\Omega,\Omega^c}| = I\left(H_{\Omega,\Omega^c} X_\Omega + Z_{\Omega^c}; X_\Omega\right), \tag{73}$$

where $X_v \sim$ i.i.d. $\mathcal{CN}(0, 1)$. Throughout the rest of this proof, we will drop the subscripts Ω and Ω^c for convenience. It is shown in the proof of Theorem 4.1 in [2] that if we restrict X_v to only the fractional part, the loss in mutual information can be bounded. We present a quick sketch here for the sake of completeness. Let $\bar{X}_v \stackrel{\text{def}}{=} X_v - [X_v]$ denote the fractional part. Then

$$\begin{aligned} I\left(HX + Z; X\right) &\leq I\left(H\bar{X} + Z, H\left[X\right]\right) && (74) \\ &\leq I(H\bar{X} + Z; \bar{X}) + H(H\left[X\right]) && (75) \\ &\leq I(H\bar{X} + Z; \bar{X}) + \sum_{v \in \Omega} H([X]) && (76) \\ &\leq I(H\bar{X} + Z; \bar{X}) + 4|\Omega|. && (77) \end{aligned}$$

Note that $\bar{X}_v \in [-1/2, 1/2)$.

2. Next, from [1](Lemma 7.2), it follows that

$$I(H\bar{X} + Z; \bar{X}) \leq I(\left[H\bar{X}\right]; \bar{X}) + 19|\Omega^c|. \tag{78}$$

Here $\left[H\bar{X}\right]$ corresponds to the output of the DSN when $\bar{X}$ is the input, however $\bar{X}$ still takes values in a continuous space and is not a permissible input in the DSN.

3. Since the reduced channel given by $Y = \left[H\bar{X}\right]$ is deterministic and further the output Y takes values in a finite set, we can also restrict $\bar{X}_v$ to take values from a finite set within $[-1/2, 1/2)$ without any loss of mutual information. Let us call this vector X^{DSN},

$$I(\left[H\bar{X}\right]; \bar{X}) = I(\left[HX^{\text{DSN}}\right]; X^{\text{DSN}}). \tag{79}$$

We have therefore showed that

$$\log|I + H_{\Omega,\Omega^c} H^*_{\Omega,\Omega^c}| \leq I\left(Y^{\text{DSN}}_{\Omega^c}; X^{\text{DSN}}_{\Omega} | X^{\text{DSN}}_{\Omega^c}\right) + 23|\mathcal{V}|. \tag{80}$$

This completes the proof of proposition. □

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