University of Illinois
ECE544

Fall 2008
Moulin

Midterm Exam

Wednesday, October 22, 2008

Name SOLUTION SET

Score __________

Please do not turn this page until requested to do so.

Problem 1 [40 pts]. Consider the 2-D random process defined by the recursion

$$X(n_1, n_2) = X(n_1 - 1, n_2) - \frac{1}{4} X(n_1 - 2, n_2) + E(n_1, n_2), \quad n_1, n_2 \in \mathbb{Z}$$

where $E(n_1, n_2)$ are iid random variables with mean 0 and variance 1.

(a) Derive the 2-D spectral density $S_X(f_1, f_2)$ for the process X.

$$\underbrace{\left[1 - z_1^{-1} + \tfrac{1}{4} z_1^{-2}\right]}_{= (1 - \frac{1}{2} z_1^{-1})^2} X(z_1, z_2) = E(z_1, z_2)$$

$$\Rightarrow \boxed{S_X(f_1, f_2) = \frac{1}{\left|1 - \frac{1}{2} e^{-j2\pi f_1}\right|^4}}$$

(b) Derive the variance of $X(n_1, n_2)$.

The coefficients of the prediction filter are $a_1 = 1$, $a_2 = -\frac{1}{4}$

Solve the Yule-Walker equations

$$\begin{cases} r_1 = a_1 r_0 + a_2 r_1 \\ r_2 = a_1 r_1 + a_2 r_0 \\ r_0 = a_1 r_1 + a_2 r_2 + \sigma_e^2 \end{cases} \Rightarrow$$

$$r_1 = \frac{a_1}{1 - a_2} r_0 = \frac{4}{5} r_0$$

$$r_2 = \left(a_1 \tfrac{4}{5} + a_2\right) r_0 = \frac{11}{20} r_0$$

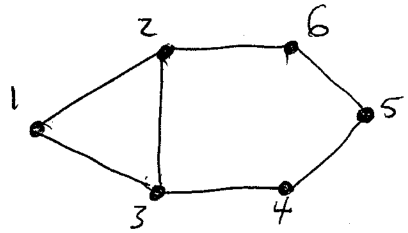
$$r_0 \underbrace{\left(1 - a_1 \tfrac{4}{5} - a_2 \tfrac{11}{20}\right)}_{= 1 - \frac{4}{5} + \frac{11}{80} = \frac{27}{80}} = 1$$

$$\Rightarrow \boxed{r_0 = \frac{80}{27}} = \mathrm{Var}\left[X(n_1, n_2)\right]$$

(c) Is the process X ergodic?

$\boxed{\text{YES}}$

Problem 2 [40 pts]. Consider the undirected graph with two cycles $1 \sim 2 \sim 3$ and $2 \sim 3 \sim 4 \sim 5 \sim 6$, and no other edges. Let $\mathbf{X} = (X_1, \cdots, X_6)$ be a random vector with pmf $p(x_1, \cdots, x_6)$ defined on this graph. Give an efficient algorithm for evaluating the conditional probability $p(x_1|x_5)$.



$$p(x_1 \ldots x_6) = \frac{1}{Z} \psi_{12}(x_1,x_2)\, \psi_{13}(x_1,x_3)\, \psi_{23}(x_2,x_3) \times \psi_{34}(x_3,x_4)\, \psi_{45}(x_4,x_5)\, \psi_{56}(x_5,x_6) \times \psi_{26}(x_2,x_6)$$

We derive $p(x_1, x_5)$ first.

Eliminating nodes 6, 2, 4, 3 in that order, we write

$$p(x_1,x_5) = \frac{1}{Z} \underbrace{\sum_{x_3} \psi_{13}(x_1,x_3) \underbrace{\sum_{x_4} \psi_{34}(x_3,x_4)\, \psi_{45}(x_4,x_5)}_{m_{35}(x_3,x_5)}}_{m_{15}(x_1,x_5)} \underbrace{\sum_{x_2} \psi_{12}(x_1,x_2) \underbrace{\sum_{x_6} \psi_{26}(x_2,x_6)\, \psi_{56}(x_5,x_6)}_{m_{25}(x_2,x_5)}}_{m'_{15}(x_1,x_5)}$$

Then

$$p(x_5) = \sum_{x_1} p(x_1,x_5)$$

and

$$p(x_1|x_5) = \frac{p(x_1,x_5)}{p(x_5)}$$

The factor Z cancels out.

Problem 3 [20 pts]. Two Poisson processes contribute to the photon count Y in a given cell of a CCD array: one with unknown parameter λ, representing incoming light, and the second with known parameter μ, representing background radiation. Give the maximum-likelihood estimate of λ given Y.

Y is Poisson distributed with parameter $\lambda + \mu$.

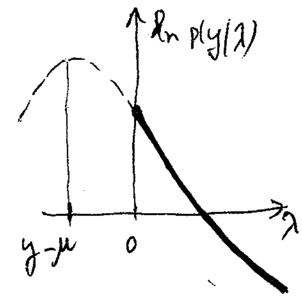
$$\Rightarrow \quad p(y|\lambda) = \frac{(\lambda+\mu)^y}{y!} e^{-(\lambda+\mu)}$$

max $\lambda \geq 0$

$$\ln p(y|\lambda) = y \ln(\lambda+\mu) \underbrace{-\lambda - \mu - \ln y!}_{\text{Constant, independent of } \lambda}$$

$$\frac{d \ln p(y|\lambda)}{d\lambda} = \frac{y}{\lambda+\mu} - 1$$

is $\begin{cases} \text{Zero for } \lambda = y-\mu \\ <0 \text{ for } \lambda < y-\mu \end{cases}$



$\Rightarrow$ Two Cases:

(I) $y > \mu \Rightarrow \hat{\lambda}_{ML} = y - \mu$

(II) $y \leq \mu \Rightarrow \hat{\lambda}_{ML} = 0$

$$\Rightarrow \boxed{\hat{\lambda}_{ML} = \max(0, y-\mu)}$$