ECE544
Statistical Image and Video Processing

`Homework #3 Solutions`

Solution to Problem 1

Consider the following HMM. The hidden sequence S_0, S_1, S_2, S_3 is a homogeneous Markov chain with alphabet $\Lambda = \{0, 1\}$ and transition probability matrix $Q = \begin{pmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{pmatrix}$. The observed variables are $\mathbf{X} = \{X_0, \cdots, X_7\} \in \Lambda^8$. The conditional pmf for $\mathbf{X}$ given $\mathbf{S}$ is given by

$$p(\mathbf{x}|\mathbf{s}) = \prod_{i=0}^{3} Q(x_{2i}|s_i) Q(x_{2i+1}|s_i).$$

Give a computationally efficient algorithm for evaluating $p(x_7|s_0)$.

Solution: write

$$p(x_7|s_0) = \sum_{s_1} Q(s_1|s_0) \sum_{s_2} Q(s_2|s_1) \sum_{s_3} Q(s_3|s_2) Q(x_7|s_3) \sum_{x_6} Q(x_6|s_3) \cdots \sum_{x_1} Q(x_1|s_0) \sum_{x_0} Q(x_0|s_0)$$

Observe that the sums over $x_0 \cdots x_6$ are all equal to 1, hence

$$\begin{aligned} p(x_7|s_0) &= \sum_{s_1} Q(s_1|s_0) \sum_{s_2} Q(s_2|s_1) \underbrace{\sum_{s_3} Q(s_3|s_2) Q(x_7|s_3)}_{m(s_2,x_7)} \\ &= \sum_{s_1} Q(s_1|s_0) \underbrace{\sum_{s_2} Q(s_2|s_1) m(s_2, x_7)}_{m(s_1,x_7)} \\ &= \sum_{s_1} Q(s_1|s_0) m(s_1, x_7) \end{aligned}$$

i.e., $p(x_7|s_0)$ is computed by successive evaluation of the above messages.

Solution to Problem 2

The global orientation of the hand is represented by a variable O, all fingers have prespecified lengths, and the angle subtended by fingers 1 and 5 (the pinkie and the thumb) is denoted by A. The orientation of the five fingers is represented by angular variables $F_1, \cdots, F_5$. In this simplified model, the hand is parameterized by the 7 variables $F_1, \cdots, F_5$, O, A.

These variables are modeled as random with the following joint pdf:

$$p(f_1, \cdots, f_5, o, a) = \frac{1}{Z}\psi_{15}(f_1, f_5|o, a)\psi_{12}(f_1, f_2)\psi_{23}(f_2, f_3)\psi_{34}(f_3, f_4)\psi_{45}(f_4, f_5)p(o)p(a).$$

1. A graphical representation of this model is given below.

   The marginal $p(f_3)$ may be evaluated by eliminating o, a, f_1, f_2, f_5, f_4 in that order:

$$\begin{aligned} p(f_3) &= \frac{1}{Z}\sum_{f_4}\psi_{34}(f_3, f_4)\sum_{f_5}\psi_{45}(f_4, f_5)\sum_{f_2}\psi_{23}(f_2, f_3)\sum_{f_1}\psi_{12}(f_1, f_2) \\ &\qquad \times\sum_{o,a}\psi_{15}(f_1, f_5|o, a)p(o)p(a). \end{aligned}$$

2. This model appears physiologically plausible because muscle configuration controls the local interactions between adjacent fingers, as you may verify experimentally. Note that the thumb is quite independent of the other fingers.

3. Assume now that $O = o^*$ and $A = 10$ (degrees) with probability 1, and that

$$\begin{aligned} \psi_{i,i+1}(f_i, f_{i+1}) &= \exp\{f_i - f_{i+1}\}\ 1_{\{f_i < f_{i+1}\}} \ : \quad i = 1, 2, 3 \\ &= \exp\{\frac{1}{4}(f_i - f_{i+1})\}\ 1_{\{f_i < f_{i+1}\}} \ : \quad i = 4 \\ \psi_{15}(f_1, f_5|o^*, 10) &= e^{-10^6|f_1+5|-10^6|f_5-5|}. \end{aligned}$$

With high probability, F_1 and F_5 are very close to -5 and 5, respectively. We can analytically evaluate $\mathbb{E}[F_3]$ because the distributions are exponential. First we write

$$\begin{aligned} p(f_3) &= \int\int\int\int df_1\, df_2\, df_4\, df_5\, p(f_1 \cdots f_5|o^*, A = 10) \\ &= \frac{1}{4Z}\int_{-\infty}^{f_2} df_1\, e^{f_1-f_2}\, e^{-10^6|f_1+5|} \int_{-\infty}^{f_3} df_2\, e^{f_2-f_3} \int_{f_3}^{\infty} df_4\, e^{f_4-f_3} \int_{f_4}^{\infty} df_5\, e^{\frac{1}{4}(f_4-f_5)}\, e^{-10^6|f_5+5|} \\ &\approx \frac{10^{-12}}{Z}\int_{-\infty}^{f_2} df_1\, e^{f_1}\, \delta(f_1+5) \int_{-\infty}^{f_3} df_2 \int_{f_3}^{\infty} df_4\, e^{-\frac{3}{4}f_4} \int_{f_4}^{\infty} df_5\, e^{-\frac{1}{4}f_5}\, \delta(f_5-5) \\ &= \frac{10^{-12}e^{-5-\frac{1}{4}5}}{Z} \int_{-5}^{f_3} df_2 \int_{f_3}^{5} df_4\, e^{-\frac{3}{4}f_4} \\ &= \frac{\frac{4}{3}\,10^6 e^{-\frac{15}{4}}}{Z}(f_3+5)(e^{-\frac{3}{4}f_3} - e^{-\frac{15}{4}}) \\ &= \frac{(f_3+5)(e^{-\frac{3}{4}f_3} - e^{-\frac{15}{4}})}{\int_{-5}^{5}(f_3+5)(e^{-\frac{3}{4}f_3} - e^{-\frac{15}{4}})\, df_3} \end{aligned}$$

where the approximation results from approximating the pdf $p(x) = \frac{1}{2}\,10^6\exp(-10^6|x|)$ with the Dirac impulse. Next

$$\mathbb{E}[F_3] = \int f_3\,p(f_3)\,df_3 \approx \frac{\int_{-5}^{5} f_3(f_3+5)(e^{-\frac{3}{4}f_3} - e^{-\frac{15}{4}})\,df_3}{\int_{-5}^{5}(f_3+5)(e^{-\frac{3}{4}f_3} - e^{-\frac{15}{4}})\,df_3}$$

Integrating by parts or evaluating both integrals numerically, we obtain $\mathbb{E}[F_3] \approx -2.439$.