Problem 1 (Computer Experiment on AR Processes)

(a) Compute the mean and the first 4 × 4 covariance samples \( R_X(n), 0 \leq n_1, n_2 \leq 3 \) for the five images (tif format) on the Web site. This will be done by computing spatial averages and making a stationarity assumption. Discuss the validity of fits of the form \( R_X(n) = C\rho^{\|n\|} \) and \( R_X(n) = C\rho^{\|n_1\|+\|n_2\|} \).

(b) Describe a simple and computationally efficient algorithm for generating realizations of the following process: \( x(n) \) is stationary and Gaussian distributed with mean 100 and covariance \( R_X(n) = 30 * \rho^{\|n_1\|+\|n_2\|} \).

(c) Generate a realization of the process in (b), using the following values for \( \rho \): .1, .5 and .95.

Problem 2 (Binary Gibbs Random Fields) Propose an energy function that strongly favors configurations made of nearest-neighbor pairs of black pixels surrounded by white pixels, or vice-versa.

Problem 3 (Computer Experiment on Ising Model) Use Gibbs sampling to generate realizations from the Ising model

\[ P(x) = Z^{-1} e^{-\frac{1}{T}U(x)}, \quad U(x) = \sum_{i \sim j} x_i x_j \]

(where each \( x_i = \pm 1 \), and the sum is over nearest neighbors).

Use a periodic 50 × 50 lattice and equiprobable, i.i.d binary random variables as your initial distribution. Use the following values for \( T^{-1} \): .1, .6, .88, 1.5 and 10. Observe the apparent convergence of the distribution to a steady state as the number of iterations increases. Explain the aspect of typical realizations for different values of \( T \) upon convergence.

Problem 4 (Optional). Repeat Problem 3 with the 8-level Potts model.

Problem 5 (Maximum-Likelihood (ML) Parameter Estimation). Consider the GGMRF model with scale parameter \( \sigma \) and exponent \( p \). The energy function takes the form

\[ U(x) = \sum_i x_i^2 + \sum_{i \sim j} \left| \frac{x_i - x_j}{\sigma} \right|^p, \]

where \( 0 < p \leq 2 \), and the second sum is over all nearest neighbors.

1. Explain why exact ML estimation of \( \sigma \) and \( p \) is difficult.

2. Develop an approximate ML estimation method for \( \sigma \) and \( p \) and apply it to the five tif images on the Web page. Comment.