Solution to ECE544 HW2

Problem 1

(a) The mean and 4×4 correlation matrices can be estimated as follows:

$$\hat{\mu} = \frac{1}{N} \sum_n x(n)$$

$$\hat{\mathsf{R}}(k) = \frac{1}{N} \sum_n x(n)x(n+k), \quad k \in \{0, 1, 2, 3\}^2.$$

We obtain the following values for the five images considered. In each case, a least-squares fit could be used to determine whether $\rho^{\|k\|}$ (isotropic model) or $\rho^{|k_1|+|k_2|}$ (separable model) is more appropriate.

$$\text{Elaine: } \hat{\mu} = 136.3565, \quad \hat{\mathsf{R}} = \begin{pmatrix} 2121.1 & 2066.6 & 2014.5 & 1960.2 \\ 2058.4 & 2047.6 & 2000.1 & 1935.2 \\ 2025.1 & 2002.9 & 1958.2 & 1907.9 \\ 1969.4 & 1952.7 & 1931.6 & 1874.9 \end{pmatrix}$$

$$\text{Man: } \hat{\mu} = 89.0075, \quad \hat{\mathsf{R}} = \begin{pmatrix} 3355.7 & 3278.3 & 3166.9 & 3070.8 \\ 3290.1 & 3241.0 & 3143.4 & 3053.6 \\ 3200.9 & 3168.7 & 3096.9 & 3020.3 \\ 3120.1 & 3097.8 & 3044.0 & 2980.2 \end{pmatrix}$$

$$\text{Aerial: } \hat{\mu} = 180.5718, \quad \hat{\mathsf{R}} = \begin{pmatrix} 1555.9 & 1398.7 & 1151.6 & 955.8 \\ 1331.7 & 1240.8 & 1060.1 & 893.9 \\ 1041.1 & 1009.3 & 920.5 & 814.2 \\ 839.4 & 825.2 & 782.9 & 721.9 \end{pmatrix}$$

$$\text{Clock: } \hat{\mu} = 185.9803, \quad \hat{\mathsf{R}} = \begin{pmatrix} 3277.5 & 3127.5 & 2973.0 & 2841.3 \\ 3190.7 & 3068.3 & 2932.4 & 2809.9 \\ 3055.2 & 2960.7 & 2855.1 & 2751.1 \\ 2935.7 & 2856.2 & 2773.2 & 2687.1 \end{pmatrix}$$

$$\text{Couple: } \hat{\mu} = 123.1772, \quad \hat{\mathsf{R}} = \begin{pmatrix} 1604.9 & 1501.1 & 1401.5 & 1334.4 \\ 1411.9 & 1351.0 & 1277.8 & 1223.7 \\ 1261.7 & 1216.6 & 1159.6 & 1117.0 \\ 1144.7 & 1107.9 & 1059.5 & 1022.7 \end{pmatrix}$$

(b) This is a separable AR(1) process:

$$x(n_1, n_2) = 100 + y(n_1, n_2)$$

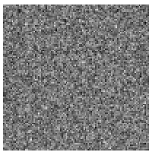
where

$$y(n_1, n_2) = 1 - \rho y(n_1 - 1, n_2) - \rho y(n_1 - 1, n_2) + \rho^2 y(n_1 - 1, n_2 - 1) + e(n_1, n_2)$$

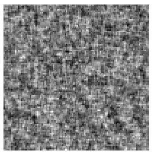
and $e(n_1, n_2)$ is white noise with mean 0 and variance $30(1 - \rho^2)^2$. Realizations of this process may be obtained by initializing the boundary of the image x (e.g., with zeroes) and recursively applying the formula above. A more efficient implementation would exploit the separability nature of the process by implementing horizontal filtering in a first step, and vertical filtering in a second step:

$$\begin{aligned} z(n_1, n_2) &= \rho z(n_1 - 1, n_2) + e(n_1, n_2) \\ y(n_1, n_2) &= \rho y(n_1, n_2 - 1) + z(n_1, n_2) \end{aligned}$$

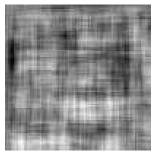
(c) For $\rho = 0.1, 0.5, 0.95$ we obtain the following realizations:



$\rho = 0.1$



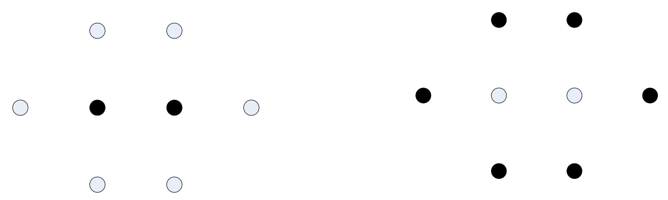
$\rho = 0.5$



$\rho = 0.95$

Problem 2

We want to design an energy function $U(x) = \sum_{\mathcal{C} \in \mathscr{C}} V_{\mathcal{C}}(x_{\mathcal{C}})$ that favors the following two types of configurations:



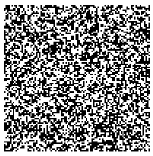
Say Black= -1 and White=1. Consider the third-order neighborhood system and let $\mathscr{C}$ be the system of all cliques of the form $\mathcal{C} = \{i, i^E, i^N, i^W, i^S\}$ and potential functions

$$V_{\mathcal{C}}(x_{\mathcal{C}}) = 1_{\{x_i s_i \neq -2\}}$$

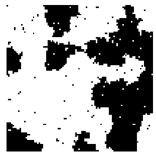
where $s_i = x_i^E + x_i^N + x_i^W + x_i^S \in \{0, \pm 2, \pm 4\}$. For the two configurations above, the energy is zero. For any checkerboard image expanded by a factor of 2 in either the horizontal or the vertical direction, the total energy will be zero.

Problem 3: Simulation of Ising model.

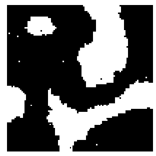
After 100 iterations of the Gibbs sampler, we obtain the following configurations:



$T^{-1} = 0.1$



$T^{-1} = 0.5$



$T^{-1} = 0.88$

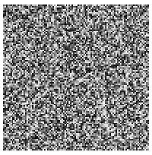
$T^{-1} = 1.5$

$T^{-1} = 10$

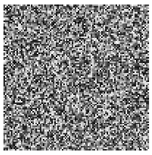
Observe that clusters become larger as the temperature parameter T decreases.

Problem 4: Simulation of 8-level Potts model.

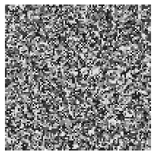
After 100 iterations of the Gibbs sampler, we obtain the following configurations:



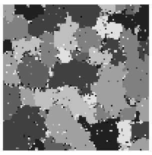
$T^{-1} = 0.1$



$T^{-1} = 0.5$



$T^{-1} = 0.88$



$T^{-1} = 1.5$

$T^{-1} = 10$

Observe that clusters become larger as the temperature parameter T decreases.

Problem 5.

The energy function takes the form

$$U(x) = \epsilon \sum_i \left(\frac{x_i}{\sigma}\right)^2 + \sum_{i \sim j} \left|\frac{x_i - x_j}{\sigma}\right|^p, \tag{1}$$

where $0 < p \leq 2$, and the second sum is over all nearest neighbors.

This problem is similar to one in a paper by Saquib, Bouman and Sauer (IEEE Trans. Image Proc., July 1998). Their results are obtained in the limit as $\epsilon \downarrow 0$. If you "neglect" the first term in (1), this amounts to taking the Saquib problem formulation.

For fixed p, the scale factor σ can be estimated as described in the lecture notes. However p is unknown, and this makes the estimation problem hard due to the dependency of the normalization constant Z upon p.

To solve our problem, one idea is to use Besag's coding method. Use two codings as described in the notes, derive the local characteristics of the MRF, and maximize the pseudo-likelihood function over p for each coding. Then average the results. All this is numerically tractable. Specifically, the local characteristics of the MRF are given by

$$p(x_i|x_{\mathcal{N}_i}) = \frac{1}{Z_i(p,\sigma)} \exp\left\{-\epsilon \left(\frac{x_i}{\sigma}\right)^2 - \sum_{j \in \mathcal{N}_i} \left|\frac{x_i - x_j}{\sigma}\right|^p\right\}$$

where the normalization constant

$$Z_i(p,\sigma) = \int_{-\infty}^{\infty} dx_i \exp\left\{-\epsilon \left(\frac{x_i}{\sigma}\right)^2 - \sum_{j \in \mathcal{N}_i} \left|\frac{x_i - x_j}{\sigma}\right|^p\right\}$$

is obtained by numerical integration (a 1-D integral). The pseudo-loglikelihood function for the first coding $\mathcal{C}_\bullet$ is given by

$$l_\bullet(p,\sigma) = \sum_{i \in \mathcal{C}_\bullet} \left[-\ln Z_i(p,\sigma) - \epsilon \left(\frac{x_i}{\sigma}\right)^2 - \sum_{j \in \mathcal{N}_i} \left|\frac{x_i - x_j}{\sigma}\right|^p\right]$$

and can be maximized over (p, σ) using numerical optimization. This yields estimates $(\hat{p}_\bullet, \hat{\sigma}_\bullet)$. Proceeding similarly with the second coding $\mathcal{C}_\circ$, we obtain estimates $(\hat{p}_\circ, \hat{\sigma}_\circ)$. The pseudo-ML estimates are finally obtained as

$$\hat{p}_{PML} = \frac{1}{2}(\hat{p}_\bullet + \hat{p}_\circ) \quad \text{and} \quad \hat{\sigma}_{PML} = \frac{1}{2}(\hat{\sigma}_\bullet + \hat{\sigma}_\circ).$$

Another idea is to use a stochastic integration method as was done in the aforementioned Saquib paper.