This first assignment is a review homework. You may have to review by yourself some material in order to get full benefit from the class.

1. **Circulant matrices**
   An $N \times N$ circulant matrix $C$ is defined by its first line, since subsequent lines are obtained by a right circular shift. Denote the first line by \( \{c_0, c_{N-1}, \ldots, c_1\} \) so that $C$ corresponds to a circular convolution with a filter having impulse response \( \{c_0, c_1, c_2, \ldots, c_{N-1}\} \).

   (a) Show that one can factorize
   \[
   C = \frac{1}{N} F_N^* A F_N
   \]
   where \( (F_N)_{k,l} = e^{-j \frac{2\pi}{N} kl} \) and $A = \text{Diag}(C[0], \ldots, C[N-1])$ with $C[k] := \sum_{l=0}^{N-1} c_l e^{-j \frac{2\pi}{N} kl}$.

   (b) Give a simple test for the singularity of $C$.

   (c) Give a formula for $\det(C)$.

   (d) Prove that $C^{-1}$ is circulant.

   (e) Show that $C_1 C_2 = C_2 C_1$ and that the result is circulant.

   *Hint:* Consult Section 2.4.8 in the textbook “Wavelet and Subband Coding” by M. Vetterli and J.Kovačević.

2. **Legendre polynomials**
   Consider the interval $[-1, 1]$ and three vectors $1$, $t$ and $t^2$ in the Hilbert space $L^2([-1, 1])$. Using Gram-Schmidt orthogonalization, find an equivalent orthonormal set.

3. **Symmetry / Antisymmetry**
   Consider the space of square-integrable real functions on the interval $[-\pi, \pi]$, $L^2([-\pi, \pi])$, and the associated orthonormal basis given by
   \[
   \left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}} \right\}, \quad n = 1, 2, \ldots
   \]
   Consider the following two subspaces: $S$ – space of symmetric functions, that is, $f(x) = f(-x)$, on $[-\pi, \pi]$, and $A$ – space of antisymmetric functions, $f(x) = -f(-x)$, on $[-\pi, \pi]$.

   (a) Show how any function $f(x)$ from $L^2([-\pi, \pi])$ can be written as $f(x) = f_s(x) + f_a(x)$, where $f_s(x) \in S$ and $f_a(x) \in A$.

   (b) Give orthonormal bases for $S$ and $A$.

   (c) Verify that $L^2([-\pi, \pi]) = S \oplus A$. 

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4. **Bonus question**: *Shannon sampling theorem*

Let $V_0$ be the sub-space of $\pi$ band-limited square integrable functions i.e.

$$V_0 := \{ f \in L^2(\mathbb{R}) \mid \hat{f}(\omega) = 0, \forall |\omega| > \pi \}$$

Let

$$\phi(t) := \text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

prove then that $\{\phi(t - n) \mid n \in \mathbb{Z}\}$ is an orthonormal basis of $V_0$. Explain the link with the Shannon sampling theorem.