Summary of Results on Two-Channel Filter Banks

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Figure 1: Two-channel (critically-sampled) filter bank.

In a two-channel filter bank as depicted in Figure 1, the following are equivalent:

1. **Perfect reconstruction (PR) condition** (i.e. \( X(z) = \hat{X}(z) \) for all input \( X(z) \)):
   \[
   \begin{align*}
   G_0(z)H_0(z) + G_1(z)H_1(z) &= 2 \\
   G_0(z)H_0(-z) + G_1(z)H_1(-z) &= 0 
   \end{align*}
   \]
   (1)

2. **Modulation-domain 1**:
   \[
   \begin{pmatrix}
   G_0(z) & G_1(z) \\
   G_0(-z) & G_1(-z) 
   \end{pmatrix}
   \begin{pmatrix}
   H_0(z) & H_0(-z) \\
   H_1(z) & H_1(-z) 
   \end{pmatrix}
   = 2I,
   \]
   (2)
   where \( I \) denotes the \( 2 \times 2 \) identity matrix.

3. **Modulation-domain 2**:
   \[
   \begin{pmatrix}
   H_0(z) & H_0(-z) \\
   H_1(z) & H_1(-z) 
   \end{pmatrix}
   \begin{pmatrix}
   G_0(z) & G_1(z) \\
   G_0(-z) & G_1(-z) 
   \end{pmatrix}
   = 2I.
   \]
   (3)

4. **Biorthogonal condition**: We can rewrite (3) as
   \[
   H_i(z)G_j(z) + H_i(-z)G_j(-z) = \delta[i-j], \quad \text{for } i, j \in \{0, 1\}.
   \]
   By denoting \( \tilde{h}[n] = h[-n] \), in the time-domain the last condition is equivalent to
   \[
   \langle \tilde{h}_i[\cdot - 2n], g_j[\cdot - 2m] \rangle = \delta[i-j]\delta[n-m] \quad \text{for } i, j \in \{0, 1\}, \text{ and } n, m \in \mathbb{Z}.
   \]
   (4)

This implies that \{\( g_0[\cdot - 2n] \), \( g_1[\cdot - 2n] \)\}_{n \in \mathbb{Z}}\ and \{\( \tilde{h}_0[\cdot - 2n] \), \( \tilde{h}_1[\cdot - 2n] \)\}_{n \in \mathbb{Z}}\ are biorthogonal bases of \( l_2(\mathbb{Z}) \), and the filter bank in Figure 1 provides a biorthogonal expansion for any signal \( x \in l_2(\mathbb{Z}) \):
   \[
   x = \sum_{n \in \mathbb{Z}} \langle \tilde{h}_0[\cdot - 2n], x \rangle g_0[\cdot - 2n] + \sum_{n \in \mathbb{Z}} \langle \tilde{h}_1[\cdot - 2n], x \rangle g_1[\cdot - 2n].
   \]
   (5)
5. Polyphase-domain 1:

\[
\begin{pmatrix}
G_{00}(z) & G_{10}(z) \\
G_{01}(z) & G_{11}(z)
\end{pmatrix}
\begin{pmatrix}
H_{00}(z) & H_{01}(z) \\
H_{10}(z) & H_{11}(z)
\end{pmatrix}
= I,
\]

(6)

where \(H_{ij}(z)\) is the \(j\)-th polyphase of the analysis filter \(H_i(z)\), i.e.

\[H_i(z) = H_{00}(z^2) + zH_{10}(z^2), \quad \text{for } i = 0, 1,\]

(7)

and \(G_{ij}(z)\) is the \(j\)-th polyphase of the synthesis filter \(G_i(z)\), i.e.

\[G_i(z) = G_{00}(z^2) + z^{-1}G_{10}(z^2), \quad \text{for } i = 0, 1.\]

(8)

Notice that analysis and synthesis filters have different polyphase decompositions.

6. Polyphase-domain 2:

\[
\begin{pmatrix}
H_{00}(z) & H_{01}(z) \\
H_{10}(z) & H_{11}(z)
\end{pmatrix}
\begin{pmatrix}
G_{00}(z) & G_{10}(z) \\
G_{01}(z) & G_{11}(z)
\end{pmatrix}
= I,
\]

(9)

A two-channel filter bank in Figure 1 that satisfies one of the above conditions is called a perfect reconstruction or biorthogonal filter bank.

A two-channel filter bank is orthogonal if it is biorthogonal and in addition we have \(h_i[n] = g_i[n]\), which is equivalent to \(H_i(z) = G_i(z^{-1})\), for \(i = 0, 1\). In that case, from (4) we see that \(\{g_i[\cdot - 2n]\}_{i \in \{0, 1\}, n \in \mathbb{Z}}\) is an orthonormal basis for \(l_2(\mathbb{Z})\) and (5) becomes an orthogonal expansion

\[x = \sum_{n \in \mathbb{Z}} \langle g_0[\cdot - 2n], x \rangle g_0[\cdot - 2n] + \sum_{n \in \mathbb{Z}} \langle g_1[\cdot - 2n], x \rangle g_1[\cdot - 2n]
= P_V x + P_W x.\]

(10)

Here \(V\) and \(W\) denote the subspaces spanned by \(\{g_0[\cdot - 2n]\}_{n \in \mathbb{Z}}\) and \(\{g_1[\cdot - 2n]\}_{n \in \mathbb{Z}}\), respectively; and \(P_S\) is the orthogonal projection onto a subspace \(S\). From (10) we see that

\[V \perp W \quad \text{and} \quad V \oplus W = l_2(\mathbb{Z}).\]

(11)

Back to perfect reconstruction filter banks, (4) implies the following biorthogonal condition on the filters \(H_0\) and \(G_0\) in the first (normally lowpass) channel:

\[H_0(z)G_0(z) + H_0(-z)G_0(-z) = 2.\]

(12)

If we consider perfect reconstruction filter banks where all filters are FIR (finite impulse response), then (2) implies that \(\det H_m(z) = \alpha^l\) where \(l\) is an odd integer. In that case, the filters in the other (normally highpass) channel are determined from the filters in the first channel (up to a scale and shift) as:

\[
\begin{align*}
H_1(z) &= z^{2k+1}G_0(-z), \\
G_1(z) &= z^{-(2k+1)}H_0(-z),
\end{align*}
\]

(13)
where $k \in \mathbb{Z}$.

Readers should quickly verify that if four filters $(H_0, H_1, G_0, G_1)$ satisfy (12) and (13) then they satisfy the perfect reconstruction condition (1). Also notice that in (13), if $H_0$ and $G_0$ are lowpass filters then $H_1$ and $G_1$ are automatically highpass filters.

We summarize with design procedures for biorthogonal and orthogonal FIR filter banks.

**Design procedure for biorthogonal FIR two-channel filter banks**

1. Design a filter $P(z)$ such that $P(z) + P(-z) = 2$, which means
   \[ P(z) = 1 + \sum_{n \text{ odd}} p[n]z^{-n}. \]

2. Factor $P(z)$ into $P(z) = H_0(z)G_0(z)$ to obtain filters of the first channel.

3. Obtain filters of the other channel $H_1$ and $G_1$ from the first one using (13).

**Design procedure for orthogonal FIR two-channel filter banks**

1. Design an autocorrelation filter $P(z) = G_0(z)G_0(z^{-1})$ such that $P(z) + P(-z) = 2$. This implies that $P(z)$ has the form:
   \[ P(z) = 1 + \sum_{n \text{ odd}, n > 0} p[n](z^{-n} + z^{n}). \]

2. Factor $P(z) = G_0(z)G_0(z^{-1})$ to obtain $G_0(z)$. That means factor $P(z)$ into roots $(\alpha, 1/\alpha)$ and assign from each pair one to $G_0(z)$.

3. Let $G_1(z) = z^{-(2k+1)}G_0(-z^{-1})$, for any $k \in \mathbb{Z}$.

4. Let $H_i(z) = G_i(z^{-1})$, for $i = 0, 1$.

So in an orthogonal filter bank, all four filters are determined (up to a scale and shift) from one filter.

**Example 1** The Haar orthogonal filter bank has:

\[
G_0(z) = \frac{1}{\sqrt{2}}(1 + z^{-1}),
\]

\[
H_0(z) = G_0(z^{-1}) = \frac{1}{\sqrt{2}}(z + 1),
\]

\[
G_1(z) = z^{-1}G_0(-z^{-1}) = \frac{1}{\sqrt{2}}(1 - z^{-1}),
\]

\[
H_1(z) = G_1(z^{-1}) = \frac{1}{\sqrt{2}}(-z + 1).
\]

Since the product $P(z) = G_0(z)G_0(z^{-1}) = \frac{1}{2}(1 + z^{-1})(1 + z)$ satisfies $P(z) + P(-z) = 2$, we can factor it differently to obtain a biorthogonal filter bank such as:

\[
G_0(z) = 1,
\]

\[
H_0(z) = \frac{1}{2}(z + 2 + z^{-1}),
\]

\[
G_1(z) = z^{-1}H_0(-z) = \frac{1}{2}(-1 + 2z^{-1} - z^{-2}),
\]

\[
H_1(z) = zG_0(-z) = z.
\]