Multidimensional and Directional Filter Banks

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Outline

1. Multidimensional filter banks
   - Sampling lattices
   - Filter design

2. Laplacian pyramid frames

3. Iterated directional filter banks
1. Multidimensional Filter Banks

**Motivation**

Looking for...

- Discrete-domain expansions for sampled data
- Structured transforms with fast algorithms
- Seamless connection with continuous-domain expansions

**Answer:** Filter banks and associated bases and frames.
Filter Banks

Wavelets and filter banks...
fundamental link between continuous and discrete domains

Mallat

analysis

synthesis

Daubechies

1. Multidimensional Filter Banks
Multidimensional Filter Banks

- **Common approach**: use 1-D techniques in a **separable** fashion

- **Limitations**:
  - very constrained filter design (separable filters)
  - only rectangular frequency partition possible
“True” Multidimensional Filter Banks

- More flexibility:
  - True multidimensional filters → directional filters
  - Sampling via lattices → discrete rotations
Sampling via Lattices

In 1D (downsample by 2):

\[ n_1 \]

In 2D (downsample by 2):

\begin{align*}
\text{1. Multidimensional Filter Banks}
\end{align*}
Multidimensional Signals and Filtering

- Discrete-time $d$-dimensional signal:
  \[ x[n], \quad n \overset{\text{def}}{=} (n_1, \ldots, n_d)^T \in \mathbb{Z}^d \]

- The $z$-transform:
  \[ X(z) = \sum_{n \in \mathbb{Z}^d} x[n] z^{-n} \quad \text{where} \quad z^n = \prod_{i=1}^{d} z_i^{n_i} \]

- Filtering with $h[n]$ yields
  \[ y[n] = x[n] \ast h[n] = \sum_{k \in \mathbb{Z}^d} x[k] h[n - k] \]
  \[ Y(z) = X(z) H(z) \]

1. Multidimensional Filter Banks
Multidimensional Sampling

Sampling operation is represented by a \( d \times d \) nonsingular integer matrix \( M \).

**Downsampling**

\[
dx_d[n] = x[Mn].
\]

**Upsampling**

\[
x_u[n] = \begin{cases} 
  x[k] & \text{if } n = Mk, \ k \in \mathbb{Z}^d \\
  0 & \text{otherwise.}
\end{cases}
\]

**Sampling lattice**

\[
LAT(M) \overset{\text{def}}{=} \{ n : n = Mk, \ k \in \mathbb{Z}^d \} = \text{all linear combinations of columns of } M \text{ with integer coefficients}
\]

**Sampling density** is equal to \( |M| \overset{\text{def}}{=} |\det M| \)
Examples of Sampling Lattices

**Separable** lattice in two dimensions:

\[
D_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},
D_2 = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}
\]

**Quincunx** lattice in two dimensions:

\[
D_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},
D_2 = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}
\]
Sampling Lattices and Matrices

**Theorem 1.** \( \text{LAT}(A) = \text{LAT}(B) \) if and only if \( A = BE \) where \( E \) is a unimodular (i.e. \( \det E = \pm 1 \)) integer matrix.

**Example:** Four basic unimodular matrices (shearing operators)

\[
R_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.
\]

**Theorem 2. [Smith form]** Any integer matrix \( M \) can be factorized as \( M = UDV \) where \( U \) and \( V \) are unimodular integer matrices and \( D \) is an integer diagonal matrices.

**Example:** For quincunx matrices

\[
Q_0 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}
\]

\[
Q_0 = R_1 D_0 R_2 = R_2 D_1 R_1 \quad \text{and} \quad Q_1 = R_0 D_0 R_3 = R_3 D_1 R_0,
\]

where \( D_0 = \text{diag}(2, 1) \) and \( D_1 = \text{diag}(1, 2) \)
Examples of Multidimensional Sampling

\[ R_3 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad Q_1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = R_0 D_0 R_3 \]
Multidimensional Sampling in Frequency-Domain (1/2)

**Downsampling** by $M$

$$X_d(\omega) = \frac{1}{|\text{det}(M)|} \sum_{k \in \mathcal{N}(M^T)} X(M^{-T}(\omega - 2\pi k)).$$

Here $\mathcal{N}(M)$ is the set of integer vectors of the form $Mt$, $t \in [0, 1)^d$. 

1. Multidimensional Filter Banks
Multidimensional Sampling in Frequency-Domain (2/2)

Upsampling by \( M \)

\[
X_u(\omega) = X(M^T \omega),
\]

\[
X_u(z) = X(z^M).
\]

Here \( z^M \overset{\text{def}}{=} (z^{m_1}, \ldots, z^{m_d})^T \), where \( m_i \) is the \( i \)-th column of \( M \).
Polyphase Representation

• Polyphase components with respect to the sampling matrix $M$:

$$x_i[n] = x[Mn + l_i]$$

where $\{l_i\}$ is the set of $|M|$ integer vectors of the form $Mt$, $t \in [0, 1)^d$

• Each polyphase component “lives” on a coset.

$$C_i \overset{\text{def}}{=} \{ m : m = Mn + l_i, \ n \in \mathbb{Z}^d \}, \quad 0 \leq i \leq |M| - 1$$

$$\bigcup_{i=0}^{\lfloor |M| - 1 \rfloor} C_i = \mathbb{Z}^d, \quad C_i \cap C_j = \emptyset, \ i \neq j.$$

• In the $z$-domain:

$$X(z) = \sum_{i=0}^{\lfloor |M| - 1 \rfloor} z^{-l_i} X_i(z^M)$$
Examples of Polyphase Representations

Separable lattice

\[ M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \]

\[ X(z_1, z_2) = X_{00}(z_1^2, z_2^2) + z_1^{-1}X_{10}(z_1^2, z_2^2) + z_2^{-1}X_{01}(z_1^2, z_2^2) + z_1^{-1}z_2^{-1}X_{11}(z_1^2, z_2^2) \]

Quincunx lattice

\[ M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \]

\[ X(z_1, z_2) = X_0(z_1z_2, z_1z_2^{-1}) + z_1^{-1}X_1(z_1z_2, z_1z_2^{-1}) \]
Polyphase-Domain Analysis

Filter and downsample

\[ x[n] \rightarrow H(z) \rightarrow (\downarrow M) \rightarrow c[n] \]

\[ C(z) = \sum_{i=0}^{\lfloor M \rfloor - 1} H_i(z)X_i(z) = H(z)x(z) \]

where \( x(z) \overset{\text{def}}{=} (X_0(z), \ldots, X_{\lfloor M \rfloor - 1}(z))^T \), and \( H(z) \overset{\text{def}}{=} (H_0(z), \ldots, H_{\lfloor M \rfloor - 1}(z)) \)

Upsample and filter

\[ c[n] \rightarrow (\uparrow M) \rightarrow G(z) \rightarrow p[n] \]

\[ P_i(z) = G_i(z)C(z) \quad \text{or} \quad p(z) = G(z)C(z) \]

where \( G(z) \overset{\text{def}}{=} (G_0(z), \ldots, G_{\lfloor M \rfloor - 1}(z))^T \), and \( p(z) \overset{\text{def}}{=} (P_0(z), \ldots, P_{\lfloor M \rfloor - 1}(z))^T \)
Filter Bank in the Polyphase-Domain

- Critically sampled: $|D| = N$

- Perfect reconstruction: $G_p(z)H_p(z) = I_{|D|}$

- Biorthogonal: Perfect reconstruction + critically sampled

- Orthogonal: $H_p(z) = G_p^T(z^{-1})$ and $G_p(z)G_p^T(z^{-1}) = I$
  (i.e. $G_p(z)$ is a paraunitary matrix)
A Reduction Theorem for Biorthogonal Filter Banks

Theorem 3. [ZhouD:04] Suppose $G(z)$ is an $N \times N$ matrix and $G_{N-1}(z)$ is its submatrix obtained by deleting the last row of $G(z)$. Suppose $H(z)$ is another $N \times N$ matrix and $H_{N-1}(z)$ is its submatrix obtained by deleting the last column of $H(z)$. Then $G(z)H(z) = I_N$ if and only if

$$G_{N-1}(z)H_{N-1}(z) = I_{N-1},$$

(1)

$$G_{N,i}(z) = \alpha(z)(-1)^{i+N} \det H_{i,N-1}(z),$$

(2)

$$H_{i,N}(z) = \alpha^{-1}(z)(-1)^{i+N} \det G_{N-1,i}(z),$$

(3)

where $\alpha(z) = \det G(z)$ is an arbitrary nonzero filter, $H_{i,N-1}(z)$ is the submatrix of $H_{N-1}(z)$ obtained by deleting its $i$th row, and $G_{N-1,i}(z)$ is the submatrix of $G_{N-1}(z)$ obtained by deleting its $i$th column.

If $G(z)$ and $H(z)$ have FIR entries, then additionally $\alpha(z) = \alpha z^k$

Corollary 1. A $N$-channel biorthogonal FIR filter bank is completely determined by its first $N - 1$ channels, and a scaled delay in the last channel.

1. Multidimensional Filter Banks
**Example: Quincunx Filter Bank**

Possible frequency partitions:

\[ \omega_0, \omega_1 \]

\[ \omega_0, (\pi, \pi) \]

Equivalent biorthogonal condition for FIR filters:

\[ H_0(z)G_0(z) + H_0(-z)G_0(-z) = 2, \quad \text{and} \]

\[ H_1(z) = z^{-k}G_0(-z), \quad G_1(z) = z^k H_0(-z). \]
Filter Design Problem

Filter design amounts to solve

\[ P(z) + P(-z) = 2, \quad \text{(easy)} \]

where

\[ P(z) = H_0(z)G_0(z) \quad \text{(hard)} \]

Fundamental problem in multi-dimensions: no factorization theorem.

How to avoid MD factorization?

One possibility: Map 1D filters to MD filters...
Design via Mapping

1. Start with a 1D solution:

\[ P^{(1D)}(z) + P^{(1D)}(-z) = 2 \quad \text{and} \quad P^{(1D)}(z) = H^{(1D)}(z)G^{(1D)}(z) \]

2. Apply a 1D to MD mapping to each filter:

\[ F^{(1D)}(z) \rightarrow F^{(MD)}(z) = F^{(1D)}(M(z)) \]

- If \( M(z) = -M(-z) \) then the perfect reconstruction condition is preserved

\[ P^{(MD)}(z) + P^{(MD)}(-z) = P^{(1D)}(M(z)) + P^{(1D)}(M(-z)) = 2 \]

- Can design the mapping to preserve and obtain other properties, like linear phase, vanishing moments, directional vanishing moments, ... (McClellan:73, TayK:93, ChenV:93, CunhaD:04)

This only works for biorthogonal filter banks...
MD Orthogonal FB Design using Cayley Transform

**Definition.** [Cayley transform] The Cayley transform of a matrix $U(z)$ is

$$H(z) = (I + U(z))^{-1}(I - U(z)).$$

**Definition.** $H(z)$ is a para-skew-Hermitian (PSH) matrix if it satisfies

$$H(z^{-1}) = -H^T(z), \quad \text{for real coefficients.}$$

**Proposition 1.** [ZhouDK:04] Suppose $U(z)$ is an $N \times N$ matrix. Then there exists at least one $N \times N$ diagonal matrix $\Lambda$ whose diagonal entries are either 1 or $-1$ such that $I + \Lambda U(z)$ is nonsingular.

**Theorem 4.** [ZhouDK:04] The Cayley transform of a paraunitary matrix is a PSH matrix. Conversely, the Cayley transform of a PSH matrix is a paraunitary matrix.
Example: Two-Channel Filter Banks

**Polyphase-domain:**

\[ U(z) = \begin{pmatrix} U_{00}(z) & U_{01}(z) \\ U_{10}(z) & U_{11}(z) \end{pmatrix} \]

The *paraunitary* condition \( U(z) U^T(z^{-1}) = I \) becomes

\[
\begin{align*}
U_{00}(z)U_{00}(z^{-1}) + U_{01}(z)U_{01}(z^{-1}) &= 1, \\
U_{00}(z)U_{10}(z^{-1}) + U_{01}(z)U_{11}(z^{-1}) &= 0, \\
U_{10}(z)U_{10}(z^{-1}) + U_{11}(z)U_{11}(z^{-1}) &= 1. 
\end{align*}
\]

\( \rightarrow \) nonlinear equations

**Cayley-domain:**

\[ H(z) = \begin{pmatrix} H_{00}(z) & H_{01}(z) \\ H_{10}(z) & H_{11}(z) \end{pmatrix} \]

The *PSH* condition \( H(z^{-1}) = -H^T(z) \) becomes

\[
\begin{align*}
H_{00}(z^{-1}) &= -H_{00}(z), \\
H_{11}(z^{-1}) &= -H_{11}(z), \\
H_{01}(z^{-1}) &= -H_{10}(z). 
\end{align*}
\]

\( \rightarrow \) linear equations

1. Multidimensional Filter Banks
Outline

1. Multidimensional filter banks

2. Laplacian pyramid frames
   - Framing pyramids
   - New reconstruction (dual frame)

3. Iterated directional filter banks
2. Laplacian Pyramid Frames

- **Reason:** avoid “frequency scrambling” due to (↓) of the HP channel.

- Laplacian pyramid as a **frame operator** → **tight frame** exists.

- New reconstruction: efficient **filter bank** for *dual frame* (pseudo-inverse).
Decomposition in the Laplacian Pyramid

\[ C(z) = H(z)x(z) \]

\[ d(z) = x(z) - G(z)H(z)x(z) = (I - G(z)H(z))x(z). \]

Combining gives

\[
\begin{pmatrix}
C(z) \\
d(z)
y(z)
\end{pmatrix}
= \begin{pmatrix}
H(z) \\
I - G(z)H(z)
\end{pmatrix}
\begin{pmatrix}
x(z)
\end{pmatrix}

2. Laplacian Pyramid Frames
Usual Reconstruction in the Laplacian Pyramid

\[ \hat{x}(z) = G(z)C(z) + d(z) \]

Or,

\[ \hat{x}(z) = \left[ \begin{array}{c} G(z) \\ I \end{array} \right] \left[ \begin{array}{c} C(z) \\ d(z) \end{array} \right] \left[ \begin{array}{c} S_1(z) \\ y(z) \end{array} \right] \]

Note that \( S_1(z)A(z) = I \) (perfect reconstruction) for any \( H(z) \) and \( G(z) \).

But... what about noisy pyramids: \( \hat{y} = y + e \) ?
Frame Analysis

• LP is a frame operator \((A)\) with redundancy.

• It admits an infinite number of left inverses.

• Let \(S\) be an arbitrary left inverse of \(A\),

\[
\hat{x} = S\hat{y} = S(y + e) = x + Se.
\]

• The optimal left inverse (minimizing \(\|S\|\)) is the pseudo-inverse (or dual frame) of \(A\):

\[
A^\dagger = (A^T A)^{-1} A^T.
\]

• If the noise is white, then among all left inverses, the pseudo-inverse minimizes the reconstruction MSE.
A Tight Frame Pyramid

Proposition. The Laplacian pyramid with orthogonal filters is a tight frame with frame bounds equal to 1.

Proof: Orthogonal condition means $G^*(z)G(z) = 1$ and $H(z) = G^*(z)$. With this, we can directly verify that

$$A^*(z)A(z) = \left(H^*(z) I - H^*(z)G^*(z)\right) \left(I - G(z)H(z)\right) = I.$$

Geometrical interpretation:

$$p[n] = \sum_{k \in \mathbb{Z}^d} \langle x[\cdot], g[\cdot - M k] \rangle g[n - M k].$$

Using the Pythagorean theorem:

$$\|x\|^2 = \|p\|^2 + \|d\|^2 = \|c\|^2 + \|d\|^2.$$
New Reconstruction for the Laplacian Pyramid

For tight frame with frame bounds 1, the pseudo-inverse of $A(z)$ is

$$A^\dagger(z) = A^T(z) = \left( I - G(z)G^T(z) \right)^T = \left( G(z) \quad I - G(z)G^T(z) \right).$$

Which leads to the optimal reconstruction:

This also works for biorthogonal filters $H$ and $G$. 

2. Laplacian Pyramid Frames
Laplacian Pyramids as an Oversampled Filter Bank

From the polyphase representation of $A(z)$ and $S(z)$

For the usual reconstruction (REC-1)

$$F_i^{[1]}(z) = z^{-k_i}, \quad \text{for } i = 0, \ldots, |M| - 1.$$  

For the new reconstruction (REC-2)

$$F_i^{[2]}(z) = z^{-k_i} - G(z)H_i(z^M), \quad \text{for } i = 0, \ldots, |M| - 1.$$  

2. Laplacian Pyramid Frames
Multilevel Laplacian Pyramids

\[ \cdots \xrightarrow{2} G \xrightarrow{2} F_0 \xrightarrow{2} F_1 \xrightarrow{2} \ldots \]

Frequency responses of equivalent synthesis filters: REC-1 and REC-2

The new reconstruction is crucial for building multiscale directional frames.

2. Laplacian Pyramid Frames
Experimental Results (1/2)

Nonlinear approximation: SNR’s of the reconstructed images from the $M$ most significant LP coefficients (after 6 levels of decomposition)

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</table>

2. Laplacian Pyramid Frames
Experimental Results (2/2)

With additive uniform white noise in $[0, 0.1]$ (non-zero mean)

usual rec.  
SNR = 6.28 dB

new rec.  
SNR = 17.42 dB

2. Laplacian Pyramid Frames
Outline

1. Multidimensional filter banks

2. Laplacian pyramid frames

3. Iterated directional filter banks
   - How to obtain directional decomposition with 2D tree-structured filter banks
   - Local directional bases
3. Iterated Directional Filter Banks

- **Feature:** division of 2-D spectrum into fine slices using tree-structured filter banks.

- **Background:** Bamberger and Smith (’92) cleverly used quincunx FB’s, modulation and shearing.

- **We propose:**
  - a simplified DFB with fan FB’s and shearing
  - use DFB to construct directional bases
Our Simplified DFB: Two Building Blocks

- Frequency splitting by the quincunx filter banks (Vetterli’84).

- Shearing by resampling

3. Iterated Directional Filter Banks
Illustration: The First Two Levels of the DFB

Using the multirate identity:

\[ \downarrow M \quad H(\omega) \quad \equiv \quad H(M^T\omega) \quad \downarrow M \]

The support configurations of equivalent filters are:

First level

Second level

Overall filters

3. Iterated Directional Filter Banks
How Frequency is Divided into Finer Direction?

Overall sampling: \[ M^{(l)}_k = [2 \cdot D_{i-2}^l] \cdot R^{s_l(k)} \]
\[ \equiv \text{separable sampling, then shearing (doesn't change lattice)} \]
Use two *separable* sampling matrices:

\[
S_k = \begin{cases} 
\begin{bmatrix} 2^{l-1} & 0 \\ 0 & 2 \end{bmatrix} & 0 \leq k < 2^{l-1} \quad \text{("near horizontal" direction)} \\
\begin{bmatrix} 2 & 0 \\ 0 & 2^{l-1} \end{bmatrix} & 2^{l-1} \leq k < 2^l \quad \text{("near vertical" direction)} 
\end{cases}
\]
**Why Critical Sampling Works?**

Frequency tiling in the MDFB:

\[
\sum_{m \in \mathbb{Z}^2} \delta_{R_k}(\omega - 2\pi S_k^{-T} m) = 1 \quad \text{for all} \quad k = 0, \ldots, 2^l - 1; \quad \omega \in \mathbb{R}^2.
\]

where \( R_k \) is the ideal frequency region for \( H_k \).
General Bases from the DFB

An \( l \)-levels DFB creates a local directional basis of \( l^2(\mathbb{Z}^2) \):

\[
\left\{ g_k^{(l)} [ \cdot - S_k^{(l)} n] \right\}_{0 \leq k < 2^l, n \in \mathbb{Z}^2}
\]

- \( G_k^{(l)} \) are directional filters:
- Sampling lattices (spatial tiling):

3. Iterated Directional Filter Banks
Example of DFB Impulse Responses

32 equivalent filters for the first half channels (basically horizontal directions) of a 6-levels DFB that use the Haar filters.
References

• Multidimensional filter banks

• Laplacian pyramid frames

• Iterated directional filter banks

• Software and downloadable papers: [www.ifp.uiuc.edu/~minhdo](http://www.ifp.uiuc.edu/~minhdo)