Contourlets: Construction and Properties

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Outline

1. Motivation

2. Discrete-domain construction using filter banks

3. Contourlets and directional multiresolution analysis

4. Contourlet approximation

5. Contourlet filter design with directional vanishing moments
What Do Image Processors Do for Living?

Compression: At 158:1 compression ratio...
What Do Image Processors Do for Living?

Denoising (restoration/filtering)

Noisy image

Clean image

1. Motivation
What Do Image Processors Do for Living?

**Feature extraction** (e.g. for content-based image retrieval)

1. Motivation
Fundamental Question: Parsimonious Representation of Visual Information

A randomly generated image
A natural image

Natural images live in a very tiny bit of the huge “image space” (e.g. $\mathbb{R}^{512 \times 512 \times 3}$)
Mathematical Foundation: Sparse Representations

Fourier, Wavelets... = construction of bases for signal expansions:

\[ f = \sum_n c_n \psi_n, \quad \text{where} \quad c_n = \langle f, \psi_n \rangle. \]

Non-linear approximation:

\[ \hat{f}_M = \sum_{n \in I_M} c_n \psi_n, \quad \text{where} \ I_M : \text{indexes of best } M \text{ components.} \]

Sparse representation: How fast \( \| f - \hat{f}_M \| \to 0 \) as \( M \to \infty \).
Wavelets and Filter Banks

1. Motivation
The Success of Wavelets

- Wavelets provide a sparse representation for piecewise smooth signals.

- Multiresolution, tree structures, fast transforms and algorithms, etc.

- Unifying theory ⇒ fruitful interaction between different fields.

1. Motivation
Fourier vs. Wavelets

Non-linear approximation: $N = 1024$ data samples; keep $M = 128$ coefficients

Using wavelets (Daubechies–4): SNR = 37.67 dB

Using Fourier (of size 1024): SNR = 22.03 dB

Approximation movie!
Is This the End of the Story?

1. Motivation
Wavelets in 2-D

• **In 1-D:** Wavelets are well adapted to *abrupt changes or singularities*.

• **In 2-D:** Separable wavelets are well adapted to *point-singularities (only)*. But, there are (mostly) *line- and curve-singularities*...
The Failure of Wavelets

Wavelets fail to capture the geometrical regularity in images and multidimensional data.

1. Motivation
Edges vs. Contours

Wavelets (with nonlinear approximations) cannot “see” the difference between these two images.

- **Edges**: image points with discontinuity
- **Contours**: edges with localized and regular direction [Zucker et al.]
Goal: Efficient Representation for Typical Images with Smooth Contours

Goal: Exploring the \textit{intrinsic geometrical structure} in natural images.

⇒ Action is at the edges!
Wavelet vs. New Scheme

For images:

- **Wavelet scheme**... see *edges* but not *smooth contours*.
- **New scheme**... requires challenging *non-separable constructions*.
And What The Nature Tells Us...

- **Human visual system:**
  - Extremely efficient: $10^7$ bits $\rightarrow$ 20-40 bits (per second).
  - Receptive fields are characterized as **localized, multiscale** and **oriented**.

- **Sparse** components of natural images (Olshausen and Field, 1996):

  16 x 16 patches from natural images
Recent Breakthrough from Harmonic Analysis: Curvelets [Candès and Donoho, 1999]

- Optimal representation for functions in $\mathbb{R}^2$ with curved singularities.
- **Key idea:** parabolic scaling relation for $C^2$ curves:

$$width \propto length^2$$

$u = u(v)$

1. Motivation
“Wish List” for New Image Representations

- Multiresolution ... successive refinement
- Localization ... both space and frequency
- Critical sampling ... correct joint sampling
- Directionality ... more directions
- Anisotropy ... more shapes

Our emphasis is on discrete framework that leads to algorithmic implementations.
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3. Contourlets and directional multiresolution analysis

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Challenge: Being Digital!

Pixelization:

Digital directions:

2. Discrete-domain construction using filter banks
Proposed Computational Framework: Contourlets

In a nutshell: contourlet transform is an efficient directional multiresolution expansion that is digital friendly!

**contourlets** = multiscale, local and directional contour segments

- Starts with a discrete-domain construction that is amenable to efficient algorithms, and then investigates its convergence to a continuous-domain expansion.

- The expansion is defined on rectangular grids $\Rightarrow$ seamless translation between the continuous and discrete worlds.
Discrete-Domain Construction using Filter Banks

**Idea:** Multiscale and Directional Decomposition

- **Multiscale step:** capture point discontinuities, followed by...
- **Directional step:** link point discontinuities into linear structures.

2. Discrete-domain construction using filter banks
Analogy: Hough Transform in Computer Vision

Challenges:

- Perfect reconstruction.
- Fixed transform with low redundancy.
- Sparse representation for images with smooth contours.
**Multiscale Decomposition using Laplacian Pyramids**

- **Reason**: avoid “frequency scrambling” due to (↓) of the HP channel.

- Laplacian pyramid as a frame operator $\rightarrow$ tight frame exists.

- New reconstruction: efficient filter bank for *dual frame* (pseudo-inverse).

2. Discrete-domain construction using filter banks
Directional Filter Banks (DFB)

• **Feature:** division of 2-D spectrum into fine slices using tree-structured filter banks.

![Diagram showing a 2D frequency spectrum with directional filter banks](image)

• **Background:** Bamberger and Smith ('92) cleverly used quincunx FB’s, modulation and shearing.

• **We propose:**
  – a simplified DFB with fan FB’s and shearing
  – use DFB to construct directional bases
Use two *separable* sampling matrices:

\[
S_k = \begin{cases} 
\begin{bmatrix} 2^{l-1} & 0 \\ 0 & 2 \end{bmatrix} & 0 \leq k < 2^{l-1} \quad ("\text{near horizontal}" \text{ direction}) \\
\begin{bmatrix} 2 & 0 \\ 0 & 2^{l-1} \end{bmatrix} & 2^{l-1} \leq k < 2^l \quad ("\text{near vertical}" \text{ direction}) 
\end{cases}
\]
General Bases from the DFB

An $l$-levels DFB creates a local directional basis of $l^2(\mathbb{Z}^2)$:

$$\left\{ g_k^{(l)}[\cdot - S_k^{(l)} n] \right\}_{0 \leq k < 2^l, n \in \mathbb{Z}^2}$$

- $G_k^{(l)}$ are directional filters:
- Sampling lattices (spatial tiling):

2. Discrete-domain construction using filter banks
Pyramidal Directional Filter Banks (PDFB)

Motivation:  
+ add multiscale into the directional filter bank  
+ improve its non-linear approximation power.

Properties:  
+ Flexible multiscale and directional representation for images  
  (can have different number of directions at each scale!)  
+ Tight frame with small redundancy (< 33%)  
+ Computational complexity: $O(N)$ for $N$ pixels.

2. Discrete-domain construction using filter banks
Wavelets vs. Contourlets

2. Discrete-domain construction using filter banks
Examples of Discrete Contourlet Transform

2. Discrete-domain construction using filter banks
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Multiresolution Analysis: Laplacian Pyramid

\[ V_{j-1} = V_j \oplus W_j, \]
\[ L^2(\mathbb{R}^2) = \bigoplus_{j \in \mathbb{Z}} W_j. \]

\( V_j \) has an orthogonal basis \( \{\phi_{j,n}\}_{n \in \mathbb{Z}^2} \), where

\[ \phi_{j,n}(t) = 2^{-j} \phi(2^{-j} t - n). \]

\( W_j \) has a tight frame \( \{\mu_{j-1,n}\}_{n \in \mathbb{Z}^2} \) where

\[ \mu_{j-1,2n+k_i} = \psi_{j,n}^{(i)}, \quad i = 0, \ldots, 3. \]
Directional Multiresolution Analysis: LP + DFB

\[ \mathcal{W}_j = \bigoplus_{k=0}^{2^{l_j}-1} \mathcal{W}_{j,k}^{(l_j)} \]

\[ \mathcal{W}_{j,k}^{(l_j)} \text{ has a tight frame } \left\{ \rho_{j,k,n}^{(l)} \right\}_{n \in \mathbb{Z}^2} \text{ where} \]

\[ \rho_{j,k,n}^{(l)}(t) = \sum_{m \in \mathbb{Z}^2} g_k^{(l)} \left[ m - S_k^{(l)} n \right] \mu_{j-1,m}(t) = \rho_{j,k}^{(l)}(t - 2^{j-1} S_k^{(l)} n). \]

3. Contourlets and directional multiresolution analysis
Contourlet Frames

Theorem (Contourlet Frames) [DoV:03].
\[
\{ \rho_{j,k,n}^{(l_j)} \}_{j \in \mathbb{Z}, \ 0 \leq k < 2^l j, \ n \in \mathbb{Z}^2}
\]
is a tight frame of \( L^2(\mathbb{R}^2) \) for finite \( l_j \).

Theorem (Connection with Filter Banks) [DoV:04]
Suppose \( x[n] = \langle f, \phi_L, n \rangle \), \( n \in \mathbb{Z}^2 \), for some function \( f \in L^2(\mathbb{R}^2) \).
Furthermore, suppose
\[
x \xrightarrow{\text{PDFB}} (a_J, d_j^{(l_j)}_{j,k})_{j=1,\ldots,J; \ k=0,\ldots,2^l j - 1}
\]
where \( a_J \) is the lowpass subband, and \( d_j^{(l_j)}_{j,k} \) are bandpass directional subbands. Then
\[
a_J[n] = \langle f, \phi_{L+J}, n \rangle
\]
\[
d_j^{(l_j)}_{j,k}[n] = \langle f, \rho_{L+j,k}^{(l_j)}, n \rangle
\]
for \( j = 1, \ldots, J; \ k = 0, \ldots, 2^l j - 1, \ n \in \mathbb{Z}^2 \).

3. Contourlets and directional multiresolution analysis
Sampling Grids of Contourlets

3. Contourlets and directional multiresolution analysis
Contourlet Features

- Defined via iterated filter banks ⇒ fast algorithms, tree structures, etc.
- Defined on rectangular grids ⇒ seamless translation between continuous and discrete worlds.
- Different contourlet kernel functions \((\rho_{j,k})\) for different directions.
- These functions are defined iteratively via filter banks.
- With FIR filters ⇒ compactly supported contourlet functions.
Contourlet Packets

- Adaptive scheme to select the “best” tree for directional decomposition.

\[ (\pi, \pi) \]

\[ (-\pi, -\pi) \]

- Contourlet packets \( \Rightarrow \) directional multiresolution elements with different shapes (aspect ratios).
  
  - They do not necessarily satisfy the parabolic relation.
  
  - They can include wavelets!
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Contourlets with Parabolic Scaling

Support size of the contourlet function $\rho_{j,k}^{l,j}$: width $\approx 2^j$ and length $\approx 2^{l_j + j}$

To satisfy the parabolic scaling (for $C^2$ curved singularities): $\text{width} \propto \text{length}^2$, simply set:

the number of directions in the PDFB is doubled at every other finer scale.

4. Contourlet approximation
Supports of Contourlet Functions

Key point: Each generation doubles spatial resolution as well as angular resolution.
**Desire:** Fast decay as contourlets turn away from the discontinuity direction

**Key:** Directional vanishing moments (DVMs)
Geometrical Intuition

At scale $2^j$ ($j \ll 0$):

width $\approx 2^j$

length $\approx 2^{j/2}$

$\# \text{directions} \approx 2^{-j/2}$

\[
|\langle f, \rho_{j,k,n} \rangle| \sim 2^{-3j/4} \cdot d_{j,k,n}^3
\]

\[
d_{j,k,n} \sim 2^j / \sin \theta_{j,k,n} \sim 2^{j/2} \tilde{k}^{-1} \quad \text{for} \quad \tilde{k} = 1, \ldots, 2^{-j/2}
\]

\[
\implies |\langle f, \rho_{j,\tilde{k},n} \rangle| \sim 2^{3j/4} \tilde{k}^{-3}
\]

4. Contourlet approximation
How Many DVMs Are Sufficient?

Sufficient if the gap to a direction with DVM:

\[ \alpha \lesssim d \sim 2^{j/2} \tilde{k}^{-1} \quad \text{for} \quad \tilde{k} = 1, \ldots, 2^{-j/2} \]

This condition can be replaced with fast decay in frequency across directions.

It is still an open question if there is an FIR filter bank that satisfies the sufficient DVM condition.
Experiments with Decay Across Directions using Near Ideal Frequency Filters

- Log$_2$(|<f, h>|) vs. Log$_2$(k)
- Disc 256 coefficients decaying plot (average)

4. Contourlet approximation
Nonlinear Approximation Rates

Under the (ideal) sufficient DVM condition

$$|\langle f, \rho_{j, \tilde{k}, n} \rangle| \sim 2^{3j/4} \tilde{k}^{-3}$$

with number of coefficients $N_{j, \tilde{k}} \sim 2^{-j/2} \tilde{k}$. Then

$$\|f - \hat{f}_M^{(\text{contourlet})}\|^2 \sim (\log M)^3 M^{-2}$$

While $\|f - \hat{f}_M^{(\text{Fourier})}\|^2 \sim O(M^{-1/2})$ and $\|f - \hat{f}_M^{(\text{wavelet})}\|^2 \sim O(M^{-1})$
Image size = $512 \times 512$. Keep $M = 4096$ coefficients.

Original image  Wavelets: PSNR = 24.34 dB  Contourlets: PSNR = 25.70 dB
Detailed Non-linear Approximations

**Wavelets**

- $M = 4$
- $M = 16$
- $M = 64$
- $M = 256$

**Contourlets**

- $M = 4$
- $M = 16$
- $M = 64$
- $M = 256$

4. Contourlet approximation
Denoising Experiments

original image

noisy image (SNR = 9.55 dB)

wavelet denoising (SNR = 13.82 dB)

contourlet denoising (SNR = 15.42 dB)

4. Contourlet approximation
So far, best-M term approximation:

\[ \hat{f}_M = \sum_{\lambda \in I_M} c_\lambda \rho_\lambda, \]  
where \( I_M \) is the set of indexes of the \( M \)-largest \( |c_\lambda| \).

For compression, additional cost required to specify \( I_M \)

- **Naive approach:** \( M \cdot \log_2 N \) bits
- **With embedded tree (wavelets):** \( M \) bits

Embedded trees for wavelets are crucial in state-of-the-art image compression (EZW,...), rate-distortion analysis (Cohen et al.), and multiscale statistical modeling (Baraniuk et al.)
Contourlet Embedded Tree Structure

Embedded tree data structure for contourlet coefficients:

successively locate the position and direction of image contours.

Since significant contourlet coefficients are organized in trees, best $M$-tree approximation (using $M$-node tree):

$$\|f - \hat{f}_{M\text{-tree}}^{(\text{contourlet})}\|^2 \approx (\log M)^3 M^{-2}$$

$$\Rightarrow D(R) \approx (\log R)^3 R^{-2}$$

4. Contourlet approximation
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Filter Bank Design Problem

\[ \rho_{j,k}^{(l)}(t) = \sum_{m \in \mathbb{Z}^2} c_k^{(l)}(m) \phi_{j-1,m}(t) \]

\( \rho_{j,k}^{(l)}(t) \) has an \( L \)-order DVM along direction \( (u_1, u_2) \)
\[ \iff C_k^{(l)}(z_1, z_2) = (1 - z_1^{u_2} \bar{z}_2^{-u_1})^L R(z_1, z_2) \]

So far: Use good frequency selectivity to approximate DVMs.

Draw back: long filters...

Next: Design short filters that lead to many DVMs as possible.

5. Contourlet filter design with directional vanishing moments
Filters with DVMs = Directional Annihilating Filters

Input image and after being filtered by a directional annihilating filter

5. Contourlet filter design with directional vanishing moments
Perfect Reconstruction Two-Channel FBs with DVMs

Filter bank with order-$L$ horizontal or vertical DVM:

Filter design amounts to solve

\[ P(z) + P(-z) = 2, \]

where \( P(z) = H_0(z)G_0(z) = (1 - z_1)^L R(z). \)

To get DVMs at other directions: Shearing or change of variables

5. Contourlet filter design with directional vanishing moments
Complete Characterization

**Proposition** [Cunha-D., 04]. Any FIR solution of

\[(1 - z_1)^L R(z) + (1 + z_1)^L R(-z) = 2\]  \hspace{1cm} (1)

can be written as

\[R(z) = A_L(z_1) + (1 + z_1)^L B(z)\]

where \(A_L(z_1)\) is the minimum degree 1-D polynomial in \(z_1\) that solves (1)

\[A_L(z_1) = \sum_{i=0}^{L-1} \binom{L + i - 1}{L - 1} 2^{-(L+i-1)}(1 - z_1)^i,\]

and \(B(z)\) satisfy \(B(z) + B(-z) = 0\).

**Proof.** Applying the Bezout theorem for each \(z_2\).
Design via Mapping (Cunha-D., 2004)

To avoid 2D factorization...

1. Start with a 1-D solution:

\[ P^{(1D)}(z) + P^{(1D)}(-z) = 2 \quad \text{and} \quad P^{(1D)}(z) = H^{(1D)}_0(z)G^{(1D)}_0(z) \]

2. Apply a 1-D to 2-D mapping to each filter:

\[ F^{(1D)}(z) \rightarrow F^{(2D)}(z) = F^{(1D)}(M(z_1, z_2)) \]

- If \( M(z) = -M(-z) \) then PR is preserved

\[ P^{(2D)}(z) + P^{(2D)}(-z) = 2 \]

- Suppose \( H^{(1D)}(z) = (1 + z)^k R_1(z) \) and \( M(z) + 1 = (1 - z_1)^l m(z) \) then

\[ H^{(2D)}(M(z)) = (1 - z_1)^{kl} R_2(z) \]
How To Design the Mapping

The mapping has to satisfy

\[ M(z) = -M(-z), \quad \text{and} \]
\[ M(z) = (1 - z_1)l m(z) - 1 \]

Thus,

\[ (1 - z_1)l m(z) + (1 + z_1)l m(-z) = 2 \]

The last proposition tell us exactly how to solve this!
5. Contourlet filter design with directional vanishing moments
Directional Vanishing Moments Generated After Iteration

Different expanding rules lead to different set of directions with DVMs.
Gain by using Filters with DVMs

5. Contourlet filter design with directional vanishing moments
Summary

• Image processing relies on *prior information* about images.
  – Geometrical structure is the key!

• Strong motivation for more powerful image representations: *scale*, *space*, and *direction*.
  – New desideratum beyond wavelets: *localized direction*

• New two-dimensional *discrete framework and algorithms*:
  – Flexible *directional* and *multiresolution* image representation.
  – Effective for images with smooth contours ⇒ contourlets.

• Dream: Another fruitful interaction between harmonic analysis, computer vision, and signal processing.
References


• Software and downloadable papers: www.ifp.uiuc.edu/~minhdo