Multidimensional Multichannel FIR Deconvolution Using Gröbner Bases

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***Abstract*— We present a new method for general multidimensional multichannel deconvolution with finite impulse response (FIR) convolution and deconvolution filters using Gröbner bases. Previous work formulates the problem of multichannel FIR deconvolution as the construction of a left inverse of the convolution matrix, which is solved by numerical linear algebra. However, this approach requires the prior information of the support of deconvolution filters. Using algebraic geometry and Gröbner bases, we find necessary and sufficient conditions for the existence of exact deconvolution FIR filters and propose simple algorithms to find these deconvolution filters. The main contribution of our work is to extend the previous Gröbner basis results on multidimensional multichannel deconvolution for polynomial or causal filters to general FIR filters. The proposed algorithms obtain a set of FIR deconvolution filters with a small number of nonzero coefficients, and do not require the prior information of the support, which is desirable in the impulsive noise environment. Moreover, we provide a complete characterization of all exact deconvolution FIR filters, from which good FIR deconvolution filters under the additive white noise environment are found. Simulation results show that our approaches achieve good results under different noise settings.**

***Index Terms*— Algebraic Geometry, Deconvolution, Exact Deconvolution, Finite Impulse Response (FIR), Gröbner Bases, Multichannel, Multidimensional, Multivariate, Nullstellensatz.**

I. Introduction

The traditional single-channel deconvolution problem is well-studied [1], [2]. In general, this problem is ill-posed since the convolution output does not contain information at frequencies corresponding to the zeros of the convolution filter. Over the last decade, *multichannel* convolution formation has become feasible and common due to the lower cost of sensors and computing units. The theory and applications of multichannel deconvolution have grown rapidly, such as general deconvolution theory [3], [4], channel equalization for multiple antennas [5], multichannel image deconvolution [6]–[10], and polarimetric calibration of radars [11]. Fig. 1 shows the multichannel deconvolution setup, where the original signal is filtered by multiple convolution filters with possible additive noise. The goal is to reconstruct the original signal from the multiple filtered signals using a set of deconvolution filters. Depending on whether the convolution filters are known, the deconvolution problem has two types: standard and blind. In this paper, we focus on the standard (nonblind) case, where the convolution filters are either known or already estimated. For the blind case, we refer the readers to [5], [7]–[9].

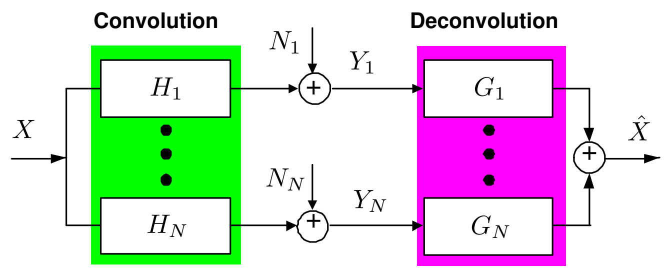

Fig. 1. An N-channel deconvolution. Original signal X is filtered by N convolution filters $\{H_1, \dots, H_N\}$ with possible additive noise. The reconstruction signal $\hat{X}$ is the sum of deconvolution outputs by N deconvolution filters $\{G_1, \dots, G_N\}$ from N inputs $\{Y_1, \dots, Y_N\}$.

Harikumar and Bresler considered the multichannel one-dimensional (1-D) and two-dimensional (2-D) *exact deconvolution* problem where both convolution and deconvolution filters are finite impulse response (FIR), and the reconstruction signal equals the original signal in the absence of additive noise [6], [12]. Giannakis and Heath also studied the multichannel 2-D exact deconvolution problem for the blind case [8]. Such FIR exact deconvolution is more computationally efficient than traditional least-square solutions. Moreover, FIR deconvolution filters limit noise propagation, which is desirable in the impulsive noise environment. They proposed an algorithm based on linear algebra to compute the deconvolution filters that requires the prior information of the support of the filters, which is unavailable in most applications. Although they provided some estimates on the support for 1-D and 2-D cases, these estimates are generally large, especially for convolution filters with different supports.

Algebraic geometry and *Gröbner bases* are powerful tools for multivariate polynomials [13], [14] and are widely used in multidimensional signal processing [15]–[18]. Rajagopal and Potter recently applied algebraic geometry to compute equalizers without the prior knowledge of the support of the deconvolution filters [11]. However, the filters they considered are only polynomial or *causal* filters, while the filters we consider here are general *FIR* filters. General FIR filters are more flexible and are used in many deconvolution applications; for example, FIR filters used in image deconvolution are typically not causal.

To apply algebraic geometry, we need to convert the FIR representation into a polynomial representation. One direct

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way to convert FIR filters into polynomial ones is multiplying the FIR filters with a monomial with high enough degree (equivalent to shifting the origin so that the filters are causal), as in [16]. However, this approach still needs the prior information or estimate of the shift of the deconvolution filters. Park, Kalker, and Vetterli proposed an algorithm transforming the FIR problem into a polynomial one [19]. However, this algorithm involves complicated transformation matrices. More comparison on different approaches in dealing with FIR filters will be provided at the end of Section III-B.

In this work, we propose a new approach for the general multidimensional multichannel FIR deconvolution problem using algebraic geometry. The key contribution is that we map the FIR deconvolution problem into a polynomial one by simply introducing a new variable. Then we propose existence conditions for FIR deconvolution filters, and simple algorithms to compute deconvolution filters based on Gröbner bases.

The rest of the paper is organized as follows. In Section II, we set up the problem and briefly introduce algebraic geometry and Gröbner bases. In Section III, we propose existence conditions for deconvolution filters, and algorithms based on Gröbner bases to compute the deconvolution filters. In Section IV, we propose a complete characterization of the deconvolution filters and discuss the optimization of deconvolution filters. Simulation results under different noise settings are given in Section V and conclusions are given in Section VI.

II. Preliminaries

A. Problem Setup

We start with notations. Throughout the paper, we refer to M as the number of dimensions or variables (for example, 2 for images and 3 for videos), and N as the number of channels. We use $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{C}$ to stand for the set of integers, the set of nonnegative integers, and the set of complex numbers, respectively. We denote sets, vectors, or matrices by boldface letters; for example, $\boldsymbol{z}$ stands for an M-dimensional complex variable $\boldsymbol{z} = [z_1, \ldots, z_M]$ in $\mathbb{C}^M$. Raising $\boldsymbol{z}$ to an M-dimensional integer vector $\boldsymbol{k} = [k_1, \ldots, k_M]$ yields $\boldsymbol{z}^{\boldsymbol{k}} = \prod_{i=1}^{M} z_i^{k_i}$ and raising $\boldsymbol{z}$ to the integer -1 yields $\boldsymbol{z}^{-1} = [z_1^{-1}, \ldots, z_M^{-1}]$. For an M-dimensional signal $x(\boldsymbol{k})$, $\boldsymbol{k} \in \mathbb{Z}^M$, its z-transform $X(\boldsymbol{z})$ is defined[1] as

$$X(\boldsymbol{z}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^M} x(\boldsymbol{k}) \boldsymbol{z}^{\boldsymbol{k}},$$

and its Fourier transform is given by $X(e^{-j\boldsymbol{\omega}})$. We denote the z-transform of signals or filters by uppercase letters, and occasionally we will suppress the variable $\boldsymbol{z}$ for simplicity. Similarly, an M-dimensional filter $h(\boldsymbol{k})$, $\boldsymbol{k} \in \mathbb{Z}^M$ can be represented by its z-transform $H(\boldsymbol{z})$. The set of $\boldsymbol{k}$ such that $h(\boldsymbol{k})$ is nonzero is called the support of $h(\boldsymbol{k})$ or $H(\boldsymbol{z})$. A filter is said to be FIR if its support is finite. If the filter is both FIR and causal, then its z-transform is a multivariate polynomial. We will refer to such filters as polynomials or polynomial filters. Fig. 2 illustrates different supports of 2-D polynomial filters and FIR filters.

[1]For convenience, we define z-transform so that causal filters are polynomials in $\boldsymbol{z}$ instead of $\boldsymbol{z}^{-1}$ in the usual definition.

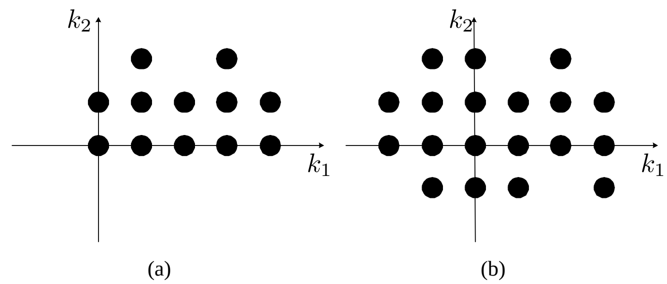

Fig. 2. Examples of supports of 2-D filters: (a) Polynomial or causal; (b) FIR.

We are interested in *exact deconvolution*, which requires that the reconstruction signal $\hat{X}$ equals the original signal X when there is no noise. One key advantage of FIR exact deconvolution is that impulsive noise does not propagate in the deconvolution output, which will be shown in Section V. We denote the convolution filters by $\{H_1, \ldots, H_N\}$ and deconvolution filters by $\{G_1, \ldots, G_N\}$ as shown in Fig. 1. Without noise, the reconstruction signal in Fig. 1 can be computed as

$$\hat{X}(\boldsymbol{z}) = \sum_{i=1}^{N} Y_i(\boldsymbol{z}) G_i(\boldsymbol{z}) = \sum_{i=1}^{N} X(\boldsymbol{z}) H_i(\boldsymbol{z}) G_i(\boldsymbol{z}).$$

Therefore, the exact deconvolution condition is equivalent to,

$$\sum_{i=1}^{N} H_i(\boldsymbol{z}) G_i(\boldsymbol{z}) = 1. \quad (1)$$

In the multichannel FIR exact deconvolution problem, the convolution filters $\{H_1, \ldots, H_N\}$ are given and FIR, and the goal is to find a set of FIR deconvolution filters $\{G_1, \ldots, G_N\}$ satisfying (1). In this paper, we will address the existence, computation, characterization, and optimization of these multidimensional FIR deconvolution filters.

Algebraic geometry and Gröbner bases are powerful tools for multivariate polynomials, but not directly for FIR filters. To apply these tools, we need to convert the condition (1) for FIR filters into a condition for polynomials. One key observation is that we can convert both $\{H_i\}$ and $\{G_i\}$ into polynomials by multiplying both sides of (1) with a monomial of high enough degree, as in [16]. Then the exact deconvolution condition for the FIR filters in (1) is equivalent to a condition for polynomial filters:

$$\sum_{i=1}^{N} H_i(\boldsymbol{z}) G_i(\boldsymbol{z}) = \boldsymbol{z}^{\boldsymbol{m}}, \quad \text{for some integer vector } \boldsymbol{m} \in \mathbb{Z}_+^M. \quad (2)$$

In applications, $\boldsymbol{z}^{\boldsymbol{m}}$ in (2) can be interpreted as a shift or delay; that is, the reconstruction signal is a shifted version of the input signal. One challenge of this conversion is that we do not know $\boldsymbol{m}$ beforehand. In this paper we will propose a novel method to solve this problem by simply introducing a new variable.

B. Algebraic Geometry and Gröbner Bases

We briefly introduce algebraic geometry and Gröbner bases. For the details, we refer readers to [13] and [20].

Suppose $\boldsymbol{k} = [k_1, \ldots, k_M]$ is a nonnegative integer vector. Then $\boldsymbol{z}^{\boldsymbol{k}}$ is a monomial with degree $\boldsymbol{k}$. A monomial ordering on M-variate polynomials is any relation $>$ on $\mathbb{Z}_+^M$, for example, the lexicographic order on $\boldsymbol{k}$. An M-variate polynomial $H(\boldsymbol{z})$ is a finite linear combination of monomials:

$$H(\boldsymbol{z}) = \sum_{\boldsymbol{k} \in \mathbb{Z}_+^M} h(\boldsymbol{k}) \boldsymbol{z}^{\boldsymbol{k}}.$$

The maximum $\boldsymbol{k}$ (in terms of a specific monomial ordering) with nonzero coefficient $h(\boldsymbol{k})$ is called the degree of $H(\boldsymbol{z})$. Similarly, the maximum $|\boldsymbol{k}|$ with nonzero coefficient $h(\boldsymbol{k})$ is called the total degree of $H(\boldsymbol{z})$, denoted by deg(H). The *ideal* generated by a polynomial set $\{H_1, \ldots, H_N\}$ is denoted as

$$\langle H_1, \ldots, H_N \rangle \stackrel{\text{def}}{=} \left\{ \sum_{i=1}^{N} H_i(\boldsymbol{z}) G_i(\boldsymbol{z}), \text{ for arbitrary polynomials } G_i(\boldsymbol{z}) \right\}. \quad (3)$$

Therefore, the polynomial deconvolution problem given in (1) can be formulated into an ideal membership problem: decide whether 1 belongs to the ideal generated by $\{H_1, \ldots, H_N\}$ and if so, find the associated $\{G_i(\boldsymbol{z})\}$ in (3).

For one dimension, the univariate membership problem is the well-known Bezout identity problem [21]. If the greatest common divisor of $\{H_1, \ldots, H_N\}$ is 1, then the Bezout identity problem has a solution. We can use the Euclidean algorithm to find the greatest common divisor and also a set of $\{G_1, \ldots, G_N\}$ [22](pp. 53-55). However, the univariate greatest common divisor criterion and Euclidean algorithm fail for multivariate polynomials, and thus cannot be directly extended to multidimensional cases. This is illustrated by the following example.

Example 1: Let $H_1(z_1, z_2) = 1 - z_1$ and $H_2(z_1, z_2) = 1 - z_2$. The greatest common divisor of $H_1(z_1, z_2)$ and $H_2(z_1, z_2)$ is 1. However, there do not exist polynomial deconvolution filters G_1 and G_2 satisfying (1). To check this, consider $[z_1, z_2] = [1, 1]$: the left side of (1) equals 0, which cannot equal the right side 1.

The multivariate membership problem can be computationally solved by Gröbner bases. Any set of polynomials can generate a Gröbner basis so that the ideal generated by the Gröbner basis is the same as the ideal generated by the given polynomial set. Specifically, given a polynomial set $\{H_1, \ldots, H_N\}$, there exist a Gröbner basis $\{B_1, \ldots, B_n\}$ and an $n \times N$ (polynomial) transform matrix $\{W_{i,j}(\boldsymbol{z})\}$ such that,

$$B_i(\boldsymbol{z}) = \sum_{j=1}^{N} W_{i,j}(\boldsymbol{z}) H_j(\boldsymbol{z}), \quad \text{for } 1 \le i \le n. \quad (4)$$

A set of polynomials may generate many Gröbner bases. However, every generated Gröbner basis can be reduced into a *reduced Gröbner basis*, which is unique not only for a given polynomial set, but also for the generated ideal. Using a procedure similar to the Euclidean algorithm, Buchberger's algorithm can be used to compute the reduced Gröbner basis and the associated transform matrix [13]. Buchberger's algorithm is implemented by many free computer algebra software such as *Macaulay2* and *Singular*, or commercial software such as *Maple* and *Mathematica*.

III. Existence and Computation

In this section, we first apply algebraic geometry and Gröbner bases to the related polynomial deconvolution problem and then extend these results to the general FIR case. For further discussion, we give the following definitions.

Definition 1: A set of polynomial filters $\{H_1, \ldots, H_N\}$ is said to be *polynomial invertible* if there exists a set of polynomial filters $\{G_1, \ldots, G_N\}$ satisfying the perfect reconstruction condition (1).

Definition 2: A set of FIR filters $\{H_1, \ldots, H_N\}$ is said to be *FIR invertible* if there exists a set of FIR filters $\{G_1, \ldots, G_N\}$ satisfying the perfect reconstruction condition (1).

Note that Definition 2 extends the exact multichannel deconvolution problem with polynomial or causal filters in Definition 1 to more general FIR filters.

A. Existence of FIR Deconvolution Filters

We first consider the existence condition for polynomial deconvolution filters. This condition can be obtained easily from the Weak Nullstellensatz theorem [6], [20].

Proposition 1: ([20], Chapter 4) Suppose $\{H_1, \ldots, H_N\}$ is a set of M-variate polynomials. Then it is polynomial invertible *if and only if* the system of equations

$$\{H_1(\boldsymbol{z}) = 0, \ \ldots, \ H_N(\boldsymbol{z}) = 0\}$$

has no solution in the complex field $\mathbb{C}^M$.

Proposition 1 provides an analytical invertibility condition on polynomial convolution filters. It can also be used as a direct test criterion for simple polynomial sets. However, Proposition 1 is not a practical criterion for general polynomial sets since it is generally difficult to test whether a system of polynomial equations has a solution in the complex field. A computational test criterion can be obtained using Gröbner bases [11], [20].

Proposition 2: ([20], Chapter 4) Suppose $\{H_1, \ldots, H_N\}$ is a set of multivariate polynomials. Then it is polynomial invertible *if and only if* its reduced Gröbner basis is $\{1\}$.

Either Proposition 1 or Proposition 2 provides a necessary and sufficient condition for the existence of *polynomial* deconvolution filters. However, for general *FIR* deconvolution filters, these conditions are not necessary. To illustrate this, we give an example.

Example 2: Let $H_1(z_1, z_2) = z_1$ and $H_2(z_1, z_2) = z_2$. They have a common zero $\boldsymbol{z} = [0, 0]$, and by Proposition 1, $\{H_1, H_2\}$ is not polynomial invertible. Also, the reduced Gröbner basis of $\{H_1, H_2\}$ is itself and hence this polynomial set does not satisfy the condition given in Proposition 2. However, let $G_1(z_1, z_2) = z_1^{-1}/2$ and $G_2(z_1, z_2) = z_2^{-1}/2$. Then they satisfy (1). In other words, $\{H_1, H_2\}$ is FIR invertible, but not polynomial invertible.

As mentioned in Section II-A, we convert the exact deconvolution condition (1) for FIR filters into the condition (2) for polynomials. Since we can shift the known FIR convolution filters $\{H_i\}$ to polynomial filters, without loss of generality, we assume that the convolution filters are polynomials in this section. The problem with solving $\{G_i\}$ in (2) from given

$\{H_i\}$ is that we do not know $\boldsymbol{m}$ beforehand. To address this problem, we propose a novel approach to generalize Proposition 1 and Proposition 2 to the general FIR case.

Definition 3: A vector is said to be *weak-zero* if at least one of its elements is zero.

Theorem 1: Suppose $\{H_1(\boldsymbol{z}),\ldots,H_N(\boldsymbol{z})\}$ is a set of multivariate polynomials. Then it is FIR invertible *if and only if* every solution of the system of equations

$$\{H_1(\boldsymbol{z})=0,\ \ldots,\ H_N(\boldsymbol{z})=0\} \tag{5}$$

is weak-zero.

Proof: For the necessary condition, suppose $\boldsymbol{s}$ is a solution of equations (5). Then substituting $\boldsymbol{z}=\boldsymbol{s}$ in (2) we have $0=\boldsymbol{s}^{\boldsymbol{m}}=s_1^{m_1}\cdots s_M^{m_M}$. Therefore, $\boldsymbol{s}$ is weak-zero.

For the sufficient condition, suppose every solution of (5) is weak-zero. Then every solution satisfies $z_1\cdots z_M=0$. By the Hilbert's Nullstellensatz Theorem [20](pp. 170-171), there exists a positive integer m such that $(z_1\cdots z_M)^m$ lies on the ideal generated by $\{H_1,\ldots,H_N\}$, which leads to (2). ■

The weak-zero is the extension of one-dimensional zero to the multidimensional cases. In the multidimensional coordinate system, a vector is weak-zero if and only if it lies on a coordinate plane. Caroenlarpnopparut and Bose used nontrivial common zero in [16], which is the complement of weak-zero. To illustrate Theorem 1, we give an example.

Example 3: Let $H_1(z_1,z_2)=z_1+z_2^2-1$ and $H_2(z_1,z_2)=z_1+z_2-1$. They have only two common zeros, $\boldsymbol{z}=[1,0]$ and $\boldsymbol{z}=[0,1]$, and both of them are weak-zero. By Theorem 1, the set $\{H_1,H_2\}$ is FIR invertible. Actually, $G_1(z_1,z_2)=-1$ and $G_2(z_1,z_2)=z_2+1$ satisfy (2) with $\boldsymbol{m}=[1,1]$.

Theorem 1 gives a sufficient and necessary condition for the FIR exact deconvolution. However, like Proposition 1, it is not a practical criteria for general sets of multivariate polynomials due to the difficulty of finding common zeros. Using Gröbner bases, we extend Proposition 2 and obtain a computational test criterion for the existence of FIR deconvolution filters.

Theorem 2: Suppose $\{H_1(\boldsymbol{z}),\ldots,H_N(\boldsymbol{z})\}$ is a set of multivariate polynomials. Then it is FIR invertible *if and only if* the reduced Gröbner basis of $\{H_1(\boldsymbol{z}),\ldots,H_N(\boldsymbol{z}),1-z_1\cdots z_{M+1}\}$ is $\{1\}$, where $\boldsymbol{z}=[z_1,\ldots,z_M]$ and z_{M+1} is a new variable.

Proof: For the necessary condition, suppose the polynomial set $\{H_1(\boldsymbol{z}),\ldots,H_N(\boldsymbol{z})\}$ is FIR invertible. Then there exists a set of polynomials $\{G_1(\boldsymbol{z}),\ldots,G_N(\boldsymbol{z})\}$ satisfying (2) for some integer vector $\boldsymbol{m}\in\mathbb{Z}_+^M$. Let m_0 be the maximum of $\{m_1,\ldots,m_M\}$. If m_0 equals 0, then by Proposition 2 the reduced Gröbner basis of the polynomial set $\{H_1(\boldsymbol{z}),\ldots,H_N(\boldsymbol{z})\}$ is $\{1\}$, and so is the set $\{H_1(\boldsymbol{z}),\ldots,H_N(\boldsymbol{z}),1-z_1\cdots z_{M+1}\}$. Otherwise, m_0 is positive. Now let

$$G_i'(\boldsymbol{z},z_{M+1})=z_{M+1}^{m_0}\prod_{j=1}^{M}z_j^{m_0-m_j}G_i(\boldsymbol{z}),\quad \text{for } i=1,\ldots,N, \tag{6}$$

and

$$G_{N+1}'(\boldsymbol{z},z_{M+1})=\sum_{i=0}^{m_0-1}\prod_{j=1}^{M+1}z_j^i. \tag{7}$$

Then, using (2), we have

$$\begin{aligned}&\sum_{i=1}^{N}H_i(\boldsymbol{z})G_i'(\boldsymbol{z},z_{M+1})+(1-\prod_{j=1}^{M+1}z_j)G_{N+1}'(\boldsymbol{z},z_{M+1})\\&=z_{M+1}^{m_0}\big(\prod_{j=1}^{M}z_j^{m_0}\big)\big(\prod_{j=1}^{M}z_j^{-m_j}\big)\sum_{i=1}^{N}H_i(\boldsymbol{z})G_i(\boldsymbol{z})\\&\quad+(1-\prod_{j=1}^{M+1}z_j)\sum_{i=0}^{m_0-1}\big(\prod_{j=1}^{M+1}z_j\big)^i\\&=z_{M+1}^{m_0}\big(\prod_{j=1}^{M}z_j^{m_0}\big)\boldsymbol{z}^{-\boldsymbol{m}}\boldsymbol{z}^{\boldsymbol{m}}+1-\big(\prod_{j=1}^{M+1}z_j\big)^{m_0}=1.\end{aligned}$$

Note that $\{G_1'(\boldsymbol{z},z_{M+1}),\ldots,G_{N+1}'(\boldsymbol{z},z_{M+1})\}$ are $(M+1)$-variate polynomials. Therefore, the set of polynomials $\{H_1(\boldsymbol{z}),\ldots,H_N(\boldsymbol{z}),1-z_1\cdots z_{M+1}\}$ is polynomial invertible and hence by Proposition 2, its reduced Gröbner basis is $\{1\}$.

For the sufficient condition, suppose the reduced Gröbner basis of $\{H_1(\boldsymbol{z}),\ldots,H_N(\boldsymbol{z}),1-z_1\cdots z_{M+1}\}$ is $\{1\}$. Then by Proposition 2, this polynomial set is polynomial invertible. In other words, there exists a polynomial set $\{G_1(\boldsymbol{z},z_{M+1}),\ldots,G_{N+1}(\boldsymbol{z},z_{M+1})\}$ satisfying (1), that is,

$$\sum_{i=1}^{N}H_i(\boldsymbol{z})G_i(\boldsymbol{z},z_{M+1})+(1-\prod_{j=1}^{M+1}z_j)G_{N+1}(\boldsymbol{z},z_{M+1})=1. \tag{8}$$

Now set $z_{M+1}=z_1^{-1}\cdots z_M^{-1}$. Then (8) becomes

$$\sum_{i=1}^{N}H_i(\boldsymbol{z})G_i(\boldsymbol{z},\prod_{j=1}^{M}z_j^{-1})=1.$$

Note that $\{G_i(\boldsymbol{z},z_1^{-1}\cdots z_M^{-1})\}$ is a set of M-variate FIR filters. Therefore, $\{H_i(\boldsymbol{z})\}$ is FIR invertible. ■

The key in this theorem is the introduction of a new variable z_{M+1} that maps the FIR deconvolution into a polynomial one. Compared to the previously proposed techniques like the ones in [19], it is considerably simpler. To illustrate Theorem 2, we give an example.

Example 4: Let $H_1(z_1,z_2)=3z_1z_2^6+z_2^6+6z_1^2z_2^3+8z_1z_2^3-3z_2^3+3z_1^3+7z_1^2+2$ and $H_2(z_1,z_2)=z_1z_2^6-2z_2^6+2z_1^2z_2^3-2z_1z_2^3+6z_2^3+z_1^3+7z_1-4$. Finding the common roots for these polynomials is difficult. However, it is easy to compute the reduced Gröbner basis of $\{H_1,H_2,1-z_1z_2z_3\}$, which is $\{1\}$. By Theorem 2, $\{H_1,H_2\}$ is FIR invertible.

Fornasini and Valcher converted the FIR inverse problem of a general FIR matrix to that of an FIR vector in [15]. However, they did not provide any computational test criterion for the FIR invertibility of FIR vectors or matrices. Theorem 2 can be easily extended to the general matrix case using the results in [15].

B. Computation of FIR Deconvolution Filters

By Proposition 2, if a set of polynomial convolution filters $\{H_1,\ldots,H_N\}$ is polynomial invertible, then its reduced Gröbner basis is $\{1\}$. In this case, we have $n=1$ and $B_1(\boldsymbol{z})=1$ in (4) and the transformation matrix $[W_{1,1},\ldots,W_{1,N}]$ gives a set of polynomial deconvolution filters. For the general FIR deconvolution problem, the constructive proof of Theorem 2 leads to the following algorithm to compute a set of FIR deconvolution filters.

Algorithm 1: The test and computational algorithm for a set of FIR deconvolution filters is given as follows.

Input: a set of M-variate FIR convolution filters $\{H_1(\boldsymbol{z}), \ldots, H_N(\boldsymbol{z})\}$.

Output: a set of FIR deconvolution filters, if it exists.

1) Multiply $\{H_i(\boldsymbol{z})\}$ by a common monomial $\boldsymbol{z}^{\boldsymbol{m}_0}$ such that $\{H_i(\boldsymbol{z})\}$ are polynomials.
2) Use the Buchberger's algorithm to compute the reduced Gröbner basis of $\{H_1(\boldsymbol{z}), \ldots, H_N(\boldsymbol{z}), 1 - z_1 \cdots z_{M+1}\}$ and the associated transform matrix $\{W_{i,j}(\boldsymbol{z}, z_{M+1})\}$ as defined in (4).
3) If the reduced Gröbner basis is $\{1\}$, then simplify $\boldsymbol{G}(\boldsymbol{z})$ given by
$$\{W_{1,1}(\boldsymbol{z}, z_1^{-1} \cdots z_M^{-1}), \ldots, W_{1,N}(\boldsymbol{z}, z_1^{-1} \cdots z_M^{-1})\} \quad (9)$$
and output $\boldsymbol{z}^{-\boldsymbol{m}_0}\boldsymbol{G}$. Otherwise, there is no solution.

Algorithm 1 generates a set of FIR deconvolution filters without the prior knowledge of the support of the deconvolution filters. In general, the resulting deconvolution filters and their support sizes depend on the choice of monomial ordering. However, all resulting filters have a small number of nonzero coefficients, due to the Buchberger's algorithm [11]. Such FIR deconvolution filters with small supports are desirable in the impulsive noise environment since they restrict noise propagation within a small region.

Example 5: Let $H_1(z_1, z_2) = z_1 + z_2^2 - 1$ and $H_2(z_1, z_2) = z_1 + z_2 - 1$. The reduced Gröbner basis of $\{H_1, H_2, 1 - z_1 z_2 z_3\}$ is $\{1\}$ and the transform matrix is $[-z_3, z_3 + z_2 z_3, 1]$. By Algorithm 1, we obtain a set of deconvolution filters $\{-z_1^{-1} z_2^{-1}, z_1^{-1} z_2^{-1} + z_1^{-1}\}$. It can be verified by exhaustive searching that this filter set has minimum number of coefficients.

Caroenlarpnopparut and Bose also explored the computation of FIR inverses in the context of two-channel filter bank design [16]. They compute the reduced Gröbner basis of $\{H_i\}$ using the Buchberger's algorithm and estimate $\boldsymbol{m}$ in (2). To find $\{G_i\}$, they use multivariate division algorithm to divide $\boldsymbol{z}^{\boldsymbol{m}}$ by the the reduced Gröbner basis of $\{H_i\}$. Park, Kalker, and Vetterli proposed a mapping method to compute FIR inverses in [19]. They first compute a square FIR invertible transform matrix and transform $\{H_i\}$ into a vector of polynomial filters $\{H_i'\}$. Then they compute the reduced Gröbner basis of $\{H_i'\}$ and obtain a polynomial inverse $\{G_i'\}$. Finally, they take the inverse transform of $\{G_i'\}$ to obtain $\{G_i\}$. Algorithm 1 generates the deconvolution filters directly from the computation of the reduced Gröbner bases. Thus, our algorithm is conceptually and computationally simpler.

Using algebraic geometry, Kollar proposed a sharp bound on the minimum shift and the minimum degree of deconvolution filters.

Proposition 3: ([23]) Suppose a set of M-variate polynomials $\{H_1, \ldots, H_N\}$ is FIR invertible. Let $d_i = \deg(H_i)$ and assume that $d_1 \geq \cdots \geq d_N$ and at most three of the d_i equal 2. Then one can find $G_1, \ldots, G_N$ and a positive integer m satisfying
$$\sum_{i=1}^{N} G_i(\boldsymbol{z}) H_i(\boldsymbol{z}) = \boldsymbol{z}^m$$
such that
$$m \leq F(M, d_1, \ldots, d_N),$$
$$\deg(G_i H_i) \leq (1 + M) \cdot F(M, d_1, \ldots, d_N),$$
where
$$F(M, d_1, \ldots, d_N) = \begin{cases} d_1 \cdots d_N & \text{if } N \leq M; \\ d_1 \cdots d_M & \text{if } N > M > 1; \\ d_1 + d_N - 1 & \text{if } N > M = 1. \end{cases}$$

C. Complexity and Numerical Stability

The main complexity of Algorithm 1 is computing Gröbner Bases. The computation complexity of Gröbner bases has been studied in a lot of literature, for example, [24], [25]. Detailed complexity analysis is beyond the scope of the paper. Experimental results show that computing Gröbner bases is very fast in practical deconvolution problems with three filters of size (that is, the number of taps) less than 36. For example, it takes less than one second on a 2 GHz PC to complete Algorithm 1 when the input is a vector of three filters of size 32.

Original Gröbner basis computation is only suitable for polynomials with rational coefficients. However, in many signal processing applications, we do not have accurate estimates of the polynomials. Therefore, numerical implementation is necessary and numerical stability is important. These issues have received a lot of attention recently in both the algebraic and the numerical analysis communities [26]–[28]. Some computer algebra software such as *Singular* have incorporated these new results and provided numerical computation of Gröbner bases. Both numerical implementation and stability analysis are beyond the scope of the paper. Experimental results show that numerical computation is usually stable. Here we show one example.

Example 6: Let $H_1(z_1, z_2) = 2z_1 z_2 + z_2 + 1$, $H_2(z_1, z_2) = z_1 + z_2 + 1$, and $H_3(z_1, z_2) = z_1^2 - 2$. The reduced Gröbner basis of $\{H_1, H_2, H_3, 1 - z_1 z_2 z_3\}$ is $\{1\}$ and the transform matrix is $[1/2 z_2 - 3/2, -z_1 z_2 + 5/2 z_1 - 1/2 z_2 + 5/2, z_2 - 5/2, 0]$. Now let $H_3'(z_1, z_2) = z_1^2 - 2.0001$. Then the reduced Gröbner basis of $\{H_1, H_2, H_3', 1 - z_1 z_2 z_3\}$ is $\{1\}$ and the transform matrix is $[0.4998 z_2 - 1.4996, -0.9996 z_1 z_2 + 2.4993 z_1 - 0.4998 z_2 + 2.4993, 0.9996 z_2 - 2.4993, 0]$.

IV. Characterization and Optimization

A. Characterization of FIR Deconvolution Filters

The set of FIR deconvolution filters is not unique for a given set of FIR convolution filters. Caroenlarpnopparut proposed a complete characterization of polynomial convolution filters based on the syzygy method in [17]. In this section, we propose a simpler characterization that uses a particular set of FIR convolution filters to characterize all sets of FIR deconvolution filters.

Using the matrix format, we express (1) as

$$\mathbf{H}^T(\boldsymbol{z})\mathbf{G}(\boldsymbol{z}) = 1, \tag{10}$$

where $\mathbf{H}(\boldsymbol{z})$ and $\mathbf{G}(\boldsymbol{z})$ are given by

$$\mathbf{H}(\boldsymbol{z}) = [H_1(\boldsymbol{z}), \ldots, H_N(\boldsymbol{z})]^T,$$
$$\mathbf{G}(\boldsymbol{z}) = [G_1(\boldsymbol{z}), \ldots, G_N(\boldsymbol{z})]^T.$$

Theorem 3: Suppose $\mathbf{H}(\boldsymbol{z})$ is a given vector of FIR convolution filters and $\mathbf{G}_p(\boldsymbol{z})$ is a vector of FIR deconvolution filters satisfying (10). Then a vector of FIR deconvolution filters $\mathbf{G}(\boldsymbol{z})$ also satisfies (10) *if and only if* $\mathbf{G}(\boldsymbol{z})$ can be written as

$$\mathbf{G}(\boldsymbol{z}) = \mathbf{G}_p(\boldsymbol{z}) + \big(\mathbf{I} - \mathbf{G}_p(\boldsymbol{z})\mathbf{H}^T(\boldsymbol{z})\big)\mathbf{S}(\boldsymbol{z}), \tag{11}$$

where $\mathbf{I}$ is an identity matrix and $\mathbf{S}(\boldsymbol{z})$ is an arbitrary vector of FIR filters.

Proof: For the sufficiency, suppose $\mathbf{G}(\boldsymbol{z})$ can be written as (11). Then

$$\mathbf{H}^T\mathbf{G} = \mathbf{H}^T(\boldsymbol{z})\mathbf{G}_p(\boldsymbol{z}) + \mathbf{H}^T(\boldsymbol{z})\big(\mathbf{I} - \mathbf{G}_p(\boldsymbol{z})\mathbf{H}^T(\boldsymbol{z})\big)\mathbf{S}(\boldsymbol{z}).$$

Since $\mathbf{G}_p(\boldsymbol{z})$ satisfies (10), $\mathbf{H}^T(\boldsymbol{z})\mathbf{G}_p(\boldsymbol{z}) = 1$. After simple manipulations, we have $\mathbf{H}^T(\boldsymbol{z})\mathbf{G}(\boldsymbol{z}) = 1$.

For the necessity, suppose $\mathbf{G}(\boldsymbol{z})$ satisfies (10). Let $\mathbf{S}(\boldsymbol{z})$ be the difference of $\mathbf{G}_p(\boldsymbol{z})$ and $\mathbf{G}(\boldsymbol{z})$:

$$\mathbf{S}(\boldsymbol{z}) = \mathbf{G}(\boldsymbol{z}) - \mathbf{G}_p(\boldsymbol{z}).$$

Since $\mathbf{G}_p(\boldsymbol{z})$ also satisfies (10), $\mathbf{H}^T(\boldsymbol{z})\mathbf{S}(\boldsymbol{z}) = 0$. Therefore,

$$\big(\mathbf{I} - \mathbf{G}_p(\boldsymbol{z})\mathbf{H}^T(\boldsymbol{z})\big)\mathbf{S}(\boldsymbol{z}) = \mathbf{S}(\boldsymbol{z}) - \mathbf{G}_p(\boldsymbol{z}) \cdot 0 = \mathbf{S}(\boldsymbol{z}),$$

which leads to (11). Moreover, since both $\mathbf{G}(\boldsymbol{z})$ and $\mathbf{G}_p(\boldsymbol{z})$ are FIR, their difference $\mathbf{S}(\boldsymbol{z})$ is also FIR. ■

Theorem 3 characterizes all FIR deconvolution filters. In this characterization, the parameter vector $S(\boldsymbol{z})$ is free, which can be used to convert the constrained optimization problem into an unconstrained one. Similar results for oversampled IIR filter banks can be found in [29]. Here we present a stronger result for FIR filters where the free vector $S(\boldsymbol{z})$ is also FIR.

B. Optimization of FIR Deconvolution Filters

Theorem 3 characterizes all sets of FIR deconvolution filters achieving the exact deconvolution. Using this characterization, we can optimize the set of deconvolution filters in terms of some criteria. One practical situation is when white noise is imposed on the convolution outputs; that is,

$$Y_i(e^{j\boldsymbol{\omega}}) = X(e^{j\boldsymbol{\omega}})H_i(e^{j\boldsymbol{\omega}}) + N_i(e^{j\boldsymbol{\omega}}). \tag{12}$$

For simplicity, we further assume that each additive noise is white noise with power density σ^2.

To compare the performance of deconvolution filters, we compute the reconstruction error $d = \hat{x} - x$ and evaluate the reconstruction mean square error (MSE) by

$$MSE = \sum_{\boldsymbol{k} \in \mathbb{Z}^M} E|d(\boldsymbol{k})|^2,$$

where E stands for the expectation. By (1) and (12), the reconstruction error (in the Fourier domain) D can be computed as

$$D(e^{j\boldsymbol{\omega}}) = \sum_{i=1}^{N} G_i(e^{j\boldsymbol{\omega}})N_i(e^{j\boldsymbol{\omega}}).$$

Since $\{N_i\}$ are independent white noises, the reconstruction MSE becomes

$$MSE = \frac{\sigma^2}{(2\pi)^M}\int_{[-\pi,\pi]^M} \widetilde{\mathbf{G}}(e^{j\boldsymbol{\omega}})\mathbf{G}(e^{j\boldsymbol{\omega}})\, d\boldsymbol{\omega} \overset{\text{def}}{=} \|\mathbf{G}(e^{j\boldsymbol{\omega}})\|, \tag{13}$$

where $\widetilde{\mathbf{G}}$ stands for the conjugate transpose of $\mathbf{G}$. Thus the optimization problem can be formulated as

$$\min_{\mathbf{G}} \|\mathbf{G}(e^{j\boldsymbol{\omega}})\|, \quad \text{such that} \;\; \mathbf{H}^T(\boldsymbol{z})\mathbf{G}(\boldsymbol{z}) = 1. \tag{14}$$

The unique optimal solution for (14) is the pseudo-inverse of $\mathbf{H}(\boldsymbol{z})$. This pseudo-inverse is FIR if and only if $\sum_{i=1}^{N} H_i(\boldsymbol{z})H_i(\boldsymbol{z}^{-1})$ is a monomial [30], which rarely happens. Generally, $\sum_{i=1}^{N} H_i(\boldsymbol{z})H_i(\boldsymbol{z}^{-1})$ is not a monomial, and hence the optimal FIR deconvolution filters do not exist in terms of the following proposition.

Proposition 4: The optimal vector of FIR exact deconvolution filters which minimizes the reconstruction mean square error in the additive white noise environment does *not* exist if $\sum_{i=1}^{N} H_i(\boldsymbol{z})H_i(\boldsymbol{z}^{-1})$ is not a monomial.

Proof: Let $\mathbf{G}^{\dagger}(\boldsymbol{z})$ denote the pseudo-inverse of $\mathbf{H}(\boldsymbol{z})$. Since $\sum_{i=1}^{N} H_i(\boldsymbol{z})H_i(\boldsymbol{z}^{-1})$ is not a monomial, $\mathbf{G}^{\dagger}(\boldsymbol{z})$ is not FIR [30].

Suppose there exists an optimal vector of FIR exact deconvolution filters, denoted by $\mathbf{G}'(\boldsymbol{z})$. Since $\mathbf{G}'(\boldsymbol{z})$ is FIR, it cannot equal $\mathbf{G}^{\dagger}(\boldsymbol{z})$. Since the pseudo-inverse gives the unique optimal solution,

$$\varepsilon = \|\mathbf{G}'(e^{j\boldsymbol{\omega}})\| - \|\mathbf{G}^{\dagger}(e^{j\boldsymbol{\omega}})\| > 0. \tag{15}$$

For $\mathbf{G}^{\dagger}(\boldsymbol{z})$, there exists an IIR filter vector $\mathbf{S}^{\dagger}(\boldsymbol{z})$ such that

$$\mathbf{G}^{\dagger}(\boldsymbol{z}) = \mathbf{G}'(\boldsymbol{z}) + \big(\mathbf{I} - \mathbf{G}'(\boldsymbol{z})\mathbf{H}^T(\boldsymbol{z})\big)\mathbf{S}^{\dagger}(\boldsymbol{z}).$$

Since $\big(\mathbf{I} - \mathbf{G}'(e^{j\boldsymbol{\omega}})\big)$ is a bounded and continuous operator, there exists a positive number δ, such that

$$\begin{aligned}&\|\mathbf{G}'(e^{j\boldsymbol{\omega}}) + \big(\mathbf{I} - \mathbf{G}'(e^{j\boldsymbol{\omega}})\mathbf{H}^T(e^{j\boldsymbol{\omega}})\big)\mathbf{S}(e^{j\boldsymbol{\omega}})\| \\ <&\|\mathbf{G}'(e^{j\boldsymbol{\omega}}) + \big(\mathbf{I} - \mathbf{G}'(e^{j\boldsymbol{\omega}})\mathbf{H}^T(e^{j\boldsymbol{\omega}})\big)\mathbf{S}^{\dagger}(e^{j\boldsymbol{\omega}})\| + \varepsilon/2,\end{aligned} \tag{16}$$

for any $\mathbf{S}(e^{j\boldsymbol{\omega}})$ satisfying

$$\|\mathbf{S}(e^{j\boldsymbol{\omega}}) - \mathbf{S}^{\dagger}(e^{j\boldsymbol{\omega}})\| < \delta. \tag{17}$$

We can find an FIR vector $\mathbf{S}^0(\boldsymbol{z})$ satisfying (17) and thus (16). Let $\mathbf{G}^0(\boldsymbol{z})$ be

$$\mathbf{G}^0(\boldsymbol{z}) = \mathbf{G}'(\boldsymbol{z}) + \big(\mathbf{I} - \mathbf{G}'(\boldsymbol{z})\mathbf{H}^T(\boldsymbol{z})\big)\mathbf{S}^0(\boldsymbol{z}). \tag{18}$$

By Theorem 3, $\mathbf{G}^0(\boldsymbol{z})$ is a vector of FIR exact deconvolution filters. Combing (15), (16), and (18), we have

$$\|\mathbf{G}^0(e^{j\boldsymbol{\omega}})\| < \|\mathbf{G}^{\dagger}(e^{j\boldsymbol{\omega}})\| + \varepsilon/2 < \|\mathbf{G}'(e^{j\boldsymbol{\omega}})\|,$$

which contradicts the assumption that $\mathbf{G}'(\boldsymbol{z})$ is an optimal vector of FIR deconvolution filters. ■

Generally, the optimal FIR exact deconvolution filters do not exist. However, if we restrict the supports of the deconvolution filters, then the optimal solution exists. If we know the supports of $\{G_i(\boldsymbol{z})\}$, then we can solve the problem using the linear algebra approach as shown in [6]. However, in general we do not know the supports of $\{G_i(\boldsymbol{z})\}$. We can always estimate these supports, but this estimate is generally large. Theorem 3 characterizes all FIR deconvolution filters with a particular set of deconvolution FIR filters $\mathbf{G}_p(\boldsymbol{z})$ and a free FIR vector $\mathbf{S}(\boldsymbol{z})$. Algorithm 1 can compute a set of FIR deconvolution filters with a small number of nonzero coefficients. If we use this particular set as $\mathbf{G}_p(\boldsymbol{z})$ and restrict the support of $\mathbf{S}(\boldsymbol{z})$, then we can find an optimal solution that minimizes the mean square error. It is difficult to analyze the resulting support of the optimal solution precisely based on the support of $\mathbf{S}(\boldsymbol{z})$. However, when the support of $\mathbf{S}(\boldsymbol{z})$ is small, the support of the optimal solution will be close to that of $\mathbf{G}_p(\boldsymbol{z})$ given in (11).

Suppose the support of $S_i(\boldsymbol{z})$ is $\boldsymbol{P}_i$ and $S_i(\boldsymbol{z})$ is parameterized as

$$S_i(\boldsymbol{z}) = \sum_{\boldsymbol{k}\in \boldsymbol{P}_i} a_{i,\boldsymbol{k}} \boldsymbol{z}^{\boldsymbol{k}}, \quad \text{for } i = 1, \ldots, N, \tag{19}$$

and the total number of $\{a_{i,\boldsymbol{k}}\}$ parameters, denoted by L. Then the vector of FIR deconvolution filters given in (11), $\mathbf{G}(\boldsymbol{z})$, can be completely parameterized by $\{a_{i,\boldsymbol{k}}\}$, and coefficients $\{g_i(\boldsymbol{k})\}$ are linear functions of $\{a_{i,\boldsymbol{k}}\}$. The goal is to minimize MSE as given in (13). By Parseval's theorem, MSE is equivalent to

$$\begin{aligned} MSE &= \frac{\sigma^2}{(2\pi)^M} \int_{[-\pi,\pi]^M} \widetilde{\mathbf{G}}(e^{j\boldsymbol{\omega}})\mathbf{G}(e^{j\boldsymbol{\omega}})\, d\boldsymbol{\omega} \\ &= \sigma^2 \sum_{i=1}^{N} \sum_{\boldsymbol{k}} |g_i(\boldsymbol{k})|^2, \end{aligned} \tag{20}$$

which implies that MSE is a continuous quadratic function of $a_{i,\boldsymbol{k}}$. We can easily find the optimal solution by setting all partial derivatives of $MSE(a_{i,\boldsymbol{k}})$ to zero, which leads to L linear equations with respect to L unknowns. A similar parametrization and optimization approach has also been used in [18] for the optimal filter bank design.

Algorithm 2: The computational algorithm for a vector of FIR exact deconvolution filters with small support is given as follows.

Input: A set of FIR convolution filters $\{H_1(\boldsymbol{z}), \ldots, H_N(\boldsymbol{z})\}$.

Output: A vector of FIR exact deconvolution filters with small support.

1) Parameterize $\mathbf{S}(\boldsymbol{z}) = [S_1(\boldsymbol{z}), \ldots, S_N(\boldsymbol{z})]^T$ in terms of (19).
2) Use Algorithm 1 to compute a vector of FIR deconvolution filters $\mathbf{G}_p(\boldsymbol{z})$.
3) Compute $\mathbf{G}(\boldsymbol{z})$ by (11) and MSE by (20).
4) Set all first-order partial derivatives of $MSE(a_{i,\boldsymbol{k}})$ to zero and solve this system of linear equations.
5) Substitute $\{a_{i,\boldsymbol{k}}\}$ in $\mathbf{G}(\boldsymbol{z})$ by the optimal solution and output $\mathbf{G}(\boldsymbol{z})$.

Algorithm 2 finds a set of FIR exact deconvolution filters that is best among the deconvolution filters with given support. This support is decided by the support of the particular set of the deconvolution filters obtained by Algorithm 1 and a specified support of free vector $\mathbf{S}(\boldsymbol{z})$. The main complexity of Algorithm 2 is using Algorithm 1 to compute Gröbner Bases and its additional complexity is negligible. The following example illustrates Algorithm 2.

Example 7: Let $H_1(z_1, z_2) = z_1 + z_2^2 - 1$ and $H_2(z_1, z_2) = z_1 + z_2 - 1$. We will use Algorithm 2 to find a best set of FIR deconvolution filters. Let the support $\boldsymbol{P} = \{(0, 0)\}$, that is, $\mathbf{S}(\boldsymbol{z})$ is a scalar vector.

1) Parameterize $\mathbf{S}(\boldsymbol{z}) = [a_1, a_2]^T$.
2) Use Algorithm 1 to compute a vector of FIR deconvolution filters $\mathbf{G}_p(\boldsymbol{z})$:
   $G_{p,1} = -z_1^{-1}z_2^{-1}$ and $G_{p,2} = z_1^{-1}z_2^{-1} + z_1^{-1}$.
3) Compute $\mathbf{G}(\boldsymbol{z})$ by (11):

$$\begin{aligned} G_1 &= a_1 + a_2 z_1^{-1} + (a_1 + a_2) z_2^{-1} \\ &\quad - (1 + a_1 - a_2) z_1^{-1} z_2^{-1} + a_1 z_1^{-1} z_2, \\ G_2 &= -a_1 + (1 + a_1) z_1^{-1} - a_2 z_2^{-1} + (1 + a_1 + a_2) \\ &\quad z_1^{-1} z_2^{-1} - (a_1 + a_2) z_1^{-1} z_2 - a_1 z_1^{-1} z_2^2. \end{aligned}$$

   Compute MSE by (20):

$$MSE(a_1, a_2) = 9a_1^2 + 4a_1a_2 + 6a_2^2 + 6a_1 + 3.$$

4) Set two partial derivatives of $MSE(a_1, a_2)$ to zero, and solve

$$\begin{cases} 9a_1 + 2a_2 &= -3 \\ a_1 + 3a_2 &= 0 \end{cases},$$

   which leads to

$$a_1 = -\frac{9}{25}, \quad a_2 = \frac{3}{25}.$$

V. Simulations

We illustrate the simulation results with both impulsive Gaussian noise and white Gaussian noise. The original 256×256 image is given in Fig. 3(a). The three convolution filters are given as

$$H_1 = \begin{pmatrix} -2 & 0 & 16 & 18 & -36 \\ 8 & -14 & 12 & \boxed{56} & -18 \\ -15 & 24 & 16 & -4 & 19 \\ 14 & 6 & -24 & -18 & -2 \\ -5 & -16 & -20 & -12 & -3 \end{pmatrix},$$

$$H_2 = \begin{pmatrix} 1 & -4 & 3 \\ -2 & \boxed{2} & 4 \\ 1 & 2 & 1 \end{pmatrix}, \quad H_3 = \begin{pmatrix} 1 & -2 & \boxed{-3} \\ -2 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix},$$

where the box represents the origin of filter supports.

In the first simulation, we show the deconvolution result with impulsive Gaussian noise, which is common in many applications such as equalization. The noise processes are added to the convolution outputs $\{Y_i\}$ independently. The impulsive noise N is defined as

$$N = \begin{cases} \sim \mathcal{N}(0, \sigma^2), & \text{with probability} \quad \alpha \\ 0, & \text{with probability} \quad (1 - \alpha) \end{cases},$$

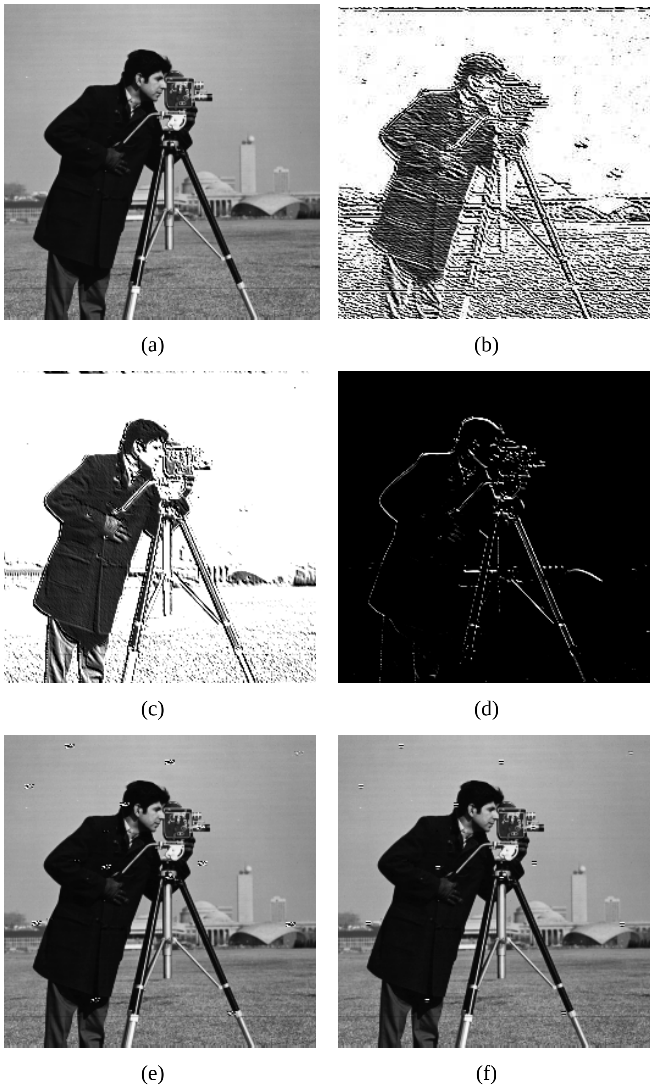

Fig. 3. (a) Original image of size 256×256. (b)-(d) Convolution outputs imposed by additive impulsive Gaussian noise (α=0.0001, MSE=51.2). Reconstruction images by the FIR deconvolution filters obtained by: (e) Linear algebra approach, MSE=30.6. (f) Our proposed approach, MSE=28.9.

where α is the occurrence probability of the impulsive noise. The convolution outputs are added with impulsive Gaussian noise and the resultant noisy outputs are shown in Fig. 3(b)-(d). In the simulation, we choose α to be 0.0001.

In this simulation, we assume that the convolution filters are known. We apply Algorithm 1 to compute a set of deconvolution filters, which are exact solutions given as

$$G_1 = \boxed{\frac{1}{40}}, \quad G_2 = \begin{pmatrix} \frac{1}{80} & \frac{3}{80} & \frac{1}{20} & \boxed{\frac{3}{20}} \\ -\frac{3}{80} & -\frac{9}{80} & -\frac{1}{8} & -\frac{1}{20} \\ \frac{3}{80} & \frac{9}{80} & \frac{9}{80} & \frac{3}{80} \\ -\frac{1}{80} & -\frac{3}{80} & -\frac{3}{80} & -\frac{1}{80} \end{pmatrix},$$

$$G_3 = \begin{pmatrix} -\frac{1}{80} & \frac{3}{80} & \frac{1}{10} & -\frac{3}{20} \\ \frac{3}{80} & -\frac{3}{80} & \boxed{\frac{3}{8}} & -\frac{1}{10} \\ -\frac{3}{80} & \frac{3}{80} & \frac{7}{80} & \frac{1}{16} \\ \frac{1}{80} & \frac{3}{80} & \frac{3}{80} & \frac{1}{80} \end{pmatrix}.$$

The first deconvolution filter is just a scalar. The size of the other two filters is 4×4. For comparison, we compute the size of deconvolution filters using the estimates in [6], which is 4×8. Then we compute a set of deconvolution filters of size 4×8 by the linear algebra approach, which gives numerical solutions. Hence, Algorithm 1 obtains deconvolution filters with smaller size. Actually, it can be verified that the obtained filters have minimum number of coefficients.

We use two sets of deconvolution filters to reconstruct the original image. The image reconstructed using the linear algebra approach is shown in Fig. 3(e), and the image reconstructed using our proposed approach is shown in Fig. 3(f). Since both sets of deconvolution filters are FIR, the impulsive noise has been isolated from propagation in both reconstruction images. However, the impulsive noises in Fig. 3(f) are smaller than those in Fig. 3(e). The reason is that the deconvolution filters obtained by our proposed approach have smaller supports than those obtained by the linear algebra approach.

In the second simulation, we show the deconvolution results with white Gaussian noise. In the experiment, the convolution outputs are imposed with white Gaussian noise with SNR 50 dB. Here the signal-to-noise ratio (SNR) is given by

$$SNR = 10 \log_{10} \left(\frac{\|Y_1\|_F^2 + \|Y_2\|_F^2 + \|Y_3\|_F^2}{3 * 256^2 \sigma^2} \right),$$

where $\| \cdot \|_F$ stands for the Frobenius norm. We first estimate the convolution filters using the 2-D EVAM Algorithm proposed in [7], [31]. The estimated convolution filters are given as[2]

$$H_1 = \begin{pmatrix} -2.0043 & -0.0088 & 16.0014 & 17.9798 & -36.0409 \\ 7.9946 & -14.0164 & 11.9783 & \boxed{56.0120} & -18.0436 \\ -14.9834 & 23.9748 & 15.9904 & -3.9696 & 18.9768 \\ 13.9945 & 6.0162 & -24.0127 & -18.0215 & -1.9952 \\ -4.9967 & -16.0024 & -20.014 & -12.0203 & -3.0109 \end{pmatrix},$$

$$H_2 = \begin{pmatrix} 1.0014 & -4.0022 & 3.0023 \\ -1.9942 & \boxed{1.9910} & 4.0075 \\ 1.0016 & 1.9983 & 1.0048 \end{pmatrix},$$

$$H_3 = \begin{pmatrix} 1.0012 & -2.0019 & \boxed{-3.0025} \\ -1.9971 & 0.0003 & 1.9978 \\ 0.9977 & 2.0017 & 1.0039 \end{pmatrix}.$$

Then we apply Algorithm 1 and compute a set of deconvolution filters, which are given as

$$G_1 = \boxed{0.02499},$$

$$G_2 = \begin{pmatrix} 0.01248 & 0.03751 & 0.0502 & \boxed{0.14996} \\ -0.03746 & -0.11253 & -0.12508 & -0.04982 \\ 0.03750 & 0.11251 & 0.11249 & 0.03748 \\ -0.01252 & -0.03754 & -0.03752 & -0.01250 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} -0.01248 & 0.03746 & 0.10013 & -0.15021 \\ 0.03742 & -0.03738 & \boxed{0.37498} & -0.09997 \\ -0.03751 & 0.03741 & 0.08752 & 0.06249 \\ 0.01257 & 0.03755 & 0.03749 & 0.01251 \end{pmatrix}.$$

This deconvolution result illustrates the numerical stability of Algorithm 1. Then we apply Algorithm 2 to compute a set of optimal deconvolution filters where we choose $\mathbf{S}(\boldsymbol{z})$

[2]We have multiplied the estimation with a scalar for comparison convenience.

to be a 3×1 scalar vector. The supports of the resulting optimal filters are 5×5, 8×8, and 8×8, respectively. For comparison, we also compute the 4×8 deconvolution filters using the linear algebra approach given in [6]. On a 2G Hz PC, Algorithm 1 and Algorithm 2 consume 0.31 and 0.32 seconds respectively, while the linear algebra approach consumes 0.13 second. Although the proposed algorithms are slower than the linear algebra approach, all of them are fast enough compared to the time-consuming 2-D EVAM algorithm, which consumes 18 seconds to estimate convolution filters.

We use these three sets of deconvolution filters to reconstruct the original image. The reconstruction images are shown in Fig. 4. The reconstruction errors result from both additive white noise and the estimation error of the convolution filters. The optimal deconvolution filters obtained by Algorithm 2 have largest supports and achieve best reconstruction quality. The reconstruction results with different noise levels are summarized in Table I and we have same observation.

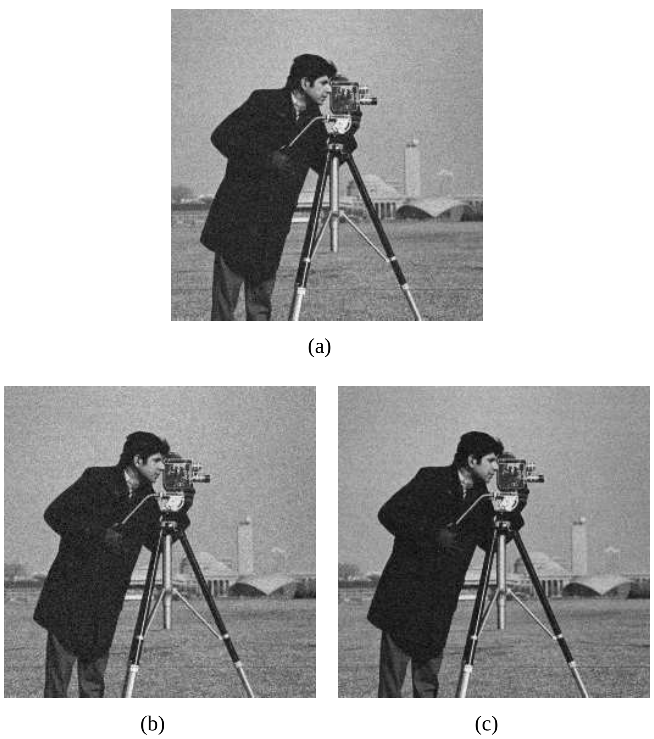

Fig. 4. The convolution outputs are imposed by additive white Gaussian noise with 50 dB. The convolution filters are estimated by the 2-D EVAM algorithm. Reconstruction images by the FIR deconvolution filters obtained by: (a) Linear algebra approach, MSE=11.54. (b) Algorithm 1, MSE=13.12. (c) Algorithm 2, MSE=10.07.

TABLE I

DECONVOLUTION PERFORMANCE (MSE) OF THE LINEAR ALGEBRA APPROACH AND PROPOSED APPROACHES UNDER THE WHITE NOISE ENVIRONMENT.

Noise SNR	Linear algebra	Algorithm 1	Algorithm 2
40 dB	36.62	41.31	33.92
45 dB	20.42	23.01	17.24
50 dB	11.54	13.12	10.07
60 dB	3.48	3.92	3.13

VI. CONCLUSION

We proposed a new method for general multidimensional multichannel FIR deconvolution using algebraic geometry and Gröbner bases. We mapped the FIR deconvolution problem into a polynomial one by simply introducing a new variable. Using algebraic geometry, we proposed sufficient and necessary conditions for FIR exact deconvolution. Then we presented simple algorithms to test invertibility and to compute a set of deconvolution FIR filters with a small number of nonzero coefficients. These algorithms do not require the prior information on the degree of deconvolution filters. We characterized the sets of deconvolution filters and used this characterization to find a best set of deconvolution filters under the additive white noise environment. The simulation results showed that the proposed algorithms achieve good deconvolution quality under different noisy environments. Our future work includes the analysis of the dependence of the filter support on the monomial ordering, and the quantifiable comparison of the supports between the linear algebra approach and the proposed algorithms.

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