Midterm Exam I
Thursday, February 26, 2009

Name ____________________________
Section: 9:00 AM 2:00 PM
Score ____________

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Please do not turn this page over until told to do so.

You may not use any books, calculators, or notes other than two sides of a 8.5” x 11” sheet of paper.

GOOD LUCK!
\[
\int_{-\infty}^{\infty} e^{-at} e^{-jwt} dt = \int_{-\infty}^{0} e^{-at} e^{-jwt} dt + \int_{0}^{\infty} e^{-at} e^{-jwt} dt \\
= \int_{-\infty}^{0} e^{t(a-j\omega)} dt + \int_{0}^{\infty} e^{-t(a+j\omega)} dt \\
= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2 + \omega^2} \quad (1 \text{ point})
\]

Problem 1 \{10 Points\}
Determine the continuous-time Fourier transform (CTFT) of \(x_1(t) = e^{-5|t|}\). Let it be denoted by \(X_1(\omega)\). The magnitude of \(X_1(\omega)\) is given in Figure 1.

1. Sketch the phase of \(X_1(\omega)\) in Figure 2.

2. Let \(X_2(\omega)\) denote the CTFT of \(x_2(t) = e^{-5|t-3|}\). Sketch the magnitude and phase of the \(X_2(\omega)\) in Figures 3 and 4.

Figure 1: Magnitude of \(X_1(\omega)\)

Figure 2: Phase of \(X_1(\omega)\)
\[
\int_{-\infty}^{\infty} e^{-a(t-3)} dt = \int_{-\infty}^{0} e^{-a(t-3)} dt + \int_{0}^{\infty} e^{-a(t-3)} dt
\]
\[
= e^{-3a} \frac{e^{3(3a-j\omega)}}{a-j\omega} + e^{3a} \frac{e^{-3a(j\omega)} \omega^2}{a+j\omega}
\]
\[
= e^{-3\omega} \cdot \frac{2a}{a^2 + \omega^2}
\]

magnitude = \frac{2a}{a^2 + \omega^2}

phase = -3\omega

If you can finish figure 3 (1 point) and 4 correctly, you get this 1 point without mathematical derivation (2 points).

Figure 3: Magnitude of \(X_2(\omega)\)

Figure 4: Phase of \(X_2(\omega)\)

\(\omega_0\) can be any number.
Problem 2 {16 Points}

The nonzero elements of a discrete-time sequence $x(n)$ are: $x[-3] = -1$, $x[-1] = 1$, $x[1] = 1$, $x[3] = 1$, $x[4] = 2$, $x[5] = 1$, $x[7] = -1$. For all other $n$, $x[n] = 0$. Calculate the following WITHOUT obtaining $X_d(\omega)$ first.

1. (a) $X_d(0)$
   
   $X_d(\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$  
   $\Rightarrow X_d(0) = \sum_{n=-\infty}^{+\infty} x[n] = 4$  
   (1 point)

   (2 points)

   (b) $\int_{-\pi}^{\pi} X_d(\omega) d\omega$
   
   $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_d(\omega) e^{+j\omega n} d\omega$  
   (1 point)

   when $n = 0$  
   $\Rightarrow x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_d(\omega) d\omega$  
   $\Rightarrow \int_{-\pi}^{\pi} X_d(\omega) d\omega = 2\pi x[0] = 0$  
   (1 point)

   (c) $X_d(\pi)$
   
   $X_d(\pi) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\pi n}$  
   $= 1 - 1 - 1 - 1 + 2 - 1 + 1 = 0$  
   (1 point)

   (2 points)

   (d) $\int_{-\pi}^{\pi} |X_d(\omega)|^2 d\omega$
   
   $= 2\pi \cdot \left[ \sum_{-\infty}^{+\infty} |x[n]|^2 \right] = 20 \pi$  
   (1 point)

   (2 points)

   (e) $\int_{-\pi}^{\pi} \left| \frac{dX_d(\omega)}{d\omega} \right|^2 d\omega$
   
   $j \frac{dX_d(\omega)}{d\omega} \leftrightarrow n x[n]$  
   (1 point)

   $\Rightarrow \int_{-\pi}^{\pi} \left| \frac{dX_d(\omega)}{d\omega} \right|^2 d\omega = 2\pi \cdot \left[ \sum_{-\infty}^{+\infty} \left| n x[n] \right|^2 \right] = 316 \pi$  
   (1 point)

2. Let $Y_d(\omega) = Re(X_d(\omega))$. Find the discrete-time sequence $y[n]$ whose DTFT is $Y_d(\omega)$.

First prove: $x^*(-n) \leftrightarrow X^*(\omega)$ (2 points)

\[
\sum_{n=-\infty}^{+\infty} x^*(-n) e^{-j\omega n} = \sum_{k=-\infty}^{+\infty} (x^*[k] e^{+j\omega k})^* = \sum_{k=-\infty}^{+\infty} (x[k] e^{-j\omega k})^* = X^*(\omega)
\]

\[
\frac{x[n] + x^*[-n]}{2} \leftrightarrow \frac{X_d(\omega) + X_d^*(\omega)}{2}
\]

(4 points) (1 point)

\[
\Rightarrow y[n] = \frac{x[n] + x^*[-n]}{2}
\]

(1 point)

$y[-7] = y[7] = \frac{1}{2}$  
$y[3] = y[-3] = 0$  
$y[-5] = y[5] = \frac{1}{2}$  
$y[4] = y[-4] = 0$  
$y[1] = y[-1] = 1$
Problem 3 (6 Points)

The continuous-time signal \( x_a(t) \) has the continuous-time Fourier transform shown in the figure below. The signal \( x_a(t) \) is sampled with sampling interval \( T \) to get the discrete-time signal \( x[n] = x_a(nT) \).

![Figure 5: \( X_a(\Omega) \)]

a. Sketch \( X_d(\omega) \) (the DTFT of \( x[n] \)) for the sampling intervals \( T = 1/200 \) and \( 1/50 \) in the corresponding frames provided. Remember to label the axes and show associated values on the axes.

b. What is the minimum sampling rate \( f_s \) (Nyquist rate) such that no aliasing will occur in sampling the continuous-time signal?

Nyquist Rate = \( \frac{100}{2} \text{ Hz} \)

or \( \frac{200}{5} \text{ Hz} \)

\( \frac{2}{2\pi} \)
Problem 4 (12 Points)

Compute the discrete-time Fourier transform (DTFT) of the following signals directly using the defining formula.

(a) \( x[n] = \sin(\frac{2\pi}{3}n) \)

\[ \sum_{n=-\infty}^{\infty} e^{-j\frac{2\pi}{3}n} - e^{-j\frac{2\pi}{3}n} \]

-1 for incorrect
-1 for not showing
-1 for not using
-1 for improper formula
-1 for not showing

(b) \( x[n] = -u[n+3] + u[n-3] \)

\[ \sum_{n=-3}^{2} e^{-j\pi n} \]

-1 for incorrect
-1 for incorrect
-1 for incorrect

(c) \( x[n] = (0.4e^{j\pi/2})^n u[n] \)

\[ \sum_{n=0}^{\infty} (0.4e^{j\pi/2})^n \]

-1 for incorrect
-1 for incorrect
-1 for incorrect

\[ \frac{1}{1 - 0.4e^{j\pi/2}} \]
Problem 5 {6 Points}

Let \( x_a(t) = \sin(7\pi t) + 0.75 \cos(5\pi t) \). Let \( \{X_m\}_{m=0}^{M-1} \) denote the order-M DFT of \( x_a(t) \).

1. Given that the analog frequency corresponding to \( X[51] \) is 3.984\( \pi \), determine the relationship between \( M \) and \( T \) where \( T \) is the sampling period.

Since 3.984\( \pi \) is positive, no need to adjust \( \omega \) by 2\( \pi \).

\[
\frac{\omega}{T} = \frac{2\pi k}{M} = \frac{2\pi k}{M T}
\]

where \( k = 51 \) and \( \omega = 3.984\pi \)

\[
MT = \frac{2\pi (51)}{3.984\pi} = \frac{102}{3.984} = 25.6024
\]

2. Given a 2 second long segment of \( x_a(t) \), how would you choose the sampling interval \( T \) to resolve the sinusoidal components and avoid aliasing? State your criterion for resolvability.

A second long segment corresponds to \( NT = 2 \)

**Criterion 1**

\[
NT \geq \frac{4\pi}{|\omega_L - \omega_R|}
\]

\[
NT \geq \frac{4\pi}{17\pi - 5\pi}
\]

\[
NT = 2
\]

\[
\Rightarrow 2 \geq 2 \checkmark
\]

**Criterion 2**

\[
NT \geq \frac{2\pi}{|\omega_L - \omega_R|}
\]

\[
NT \geq \frac{2\pi}{12\pi - 5\pi}
\]

\[
NT = 1
\]

\[
2 \geq 1 \checkmark
\]

**Nyquist**

\[
T < \frac{1}{2f_{max}}
\]

where \( f_{max} = \frac{1}{2\pi} \max(7\pi, 5\pi) \)

\[
T < \frac{1}{7}
\]

\[
f_{max} = 3.5
\]
Problem 6{13 Points}

Let \( x[n] \) denote the input and \( h[n] \) the impulse response of a linear time-invariant system. For the pairs of \( x \) and \( h \) given in parts (a)-(c), determine the output \( y[n] \). You do not need to solve parts (b) and (c) independently; use your knowledge of linearity and time invariance to minimize the work in parts (b) and (c).

Part (a)

\[ y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k] = \sum_{k=-\infty}^{\infty} a^k u[-k+1] u[n-k] \]

4. Just convolve, eqn.

3. Incorrect convolution sum

\[ y[n] = \begin{cases} \sum_{k=-\infty}^{n} a^k & n \leq -1 \\ \sum_{k=-\infty}^{-1} a^k & n > -1 \end{cases} = \begin{cases} \frac{a^{n+1}}{a-1} & n \leq -1 \\ \frac{1}{1-a} & n > -1 \end{cases} \]

Since \[ \sum_{k=m}^{n} a^k = \frac{a^{n+1}-a^m}{a-1} \] and \( |a| > 1 \).

1. Incorrect evaluation with separation of cases

Original eqn: \( y[n] = u[n-3] \)

Correct eqn: \( y[n] = a^n u[-n-1] \)

1. Incorrect evaluation with separation of cases

For each LST property to answer from (a)

Part (b) and (c)

\[ y[n] = u[n-4] * u[n] = \sum_{k=-\infty}^{\infty} y[k] u[n-k] = \begin{cases} 1 & n \leq 3 \\ 0 & n > 3 \end{cases} \]

Part (b) and (c)


Problem 7{4 Points}

Give two examples where zero-padding is useful in digital signal processing.

- Zero pad to a power of 2 for FFT

- Increase resolution in spectral analysis/DFT
Problem 8 {18 Points}

(a) Suppose you are given the 4-point discrete-time sequence \( x[n] = \{2, 1, 2, 1\} \) where the first element corresponds to \( n = 0 \).

\[ X[m] = \sum_{n=0}^{3} x[n] e^{-j\frac{2\pi}{4} nm} \]

\( m = 0 \) : \( 2 + 1 + 2 + 1 = 6 \)
\( m = 1 \) : \( 2 - j - 2 + j = 0 \) \( \Rightarrow \{6, 0, 2, 0\} \)
\( m = 2 \) : \( 2 - 1 + 2 - 1 = 2 \)
\( m = 3 \) : \( 2 + j - 2 - j = 0 \)

(b) Suppose \( x[n] \) is the sampled version of continuous-time signal \( x(t) \). Using standard notation, give the equation that relates \( x(t) \) and \( X[m] \).

\[ X[m] = \begin{cases} \frac{1}{T} \sum_{n=0}^{2} x_a \left( \frac{n m}{2T} \right) & 0 \leq m \leq 2 \\ \frac{1}{T} \sum_{n=0}^{1} x_a \left( \frac{n m}{2T} \right) & 2 < m \leq 3 \end{cases} \]

(c) What is the DFT of \( X[m] \)?

\[ \{ 4 x_{4-m} \}_{m=0}^{3} = \{ x[0] \text{ scaled by 4 and flipped except for } x[0] \} \]

\[ = \{ 2 \ 1 \ 2 \ 1 \} = \{ 8 \ 4 \ 8 \ 4 \} \]

(d) Suppose you are given another sequence \( w[n] = \{1, 2, 3, 4\} \). Let \( C[m] \) denote the cyclic convolution of \( x[n] \) and \( w[n] \). Compute the values of \( C[1] \) and \( C[3] \).

\[ C[m] = \sum_{l=0}^{3} x[l] w_{4-m+1} \] 

\( C[1] = \sum_{l=0}^{3} x[l] w_{1-l+1} = 4 + 1 + 8 + 3 = 16 \)

\( C[3] = \sum_{l=0}^{3} x[l] w_{3-l+1} = 8 + 3 + 4 + 1 = 16 \)

\( \text{Normal convolution} \quad \text{instead of cyclic} \)
Problem 9 (15 Points)

In (a)-(c), \( x[n] \) denotes the input of a system and \( y[n] \) denotes its output.

(a) \( y[n] = x[2n] \)
Is the system causal? (Yes) (No)  
(5 points)

Justify your answer:

Let \( n = 1 \Rightarrow y[1] = x[2] \)

Current \( y \) depends on future \( x \), so it is non-causal.

(b) \( y[n] = n^2 x[2n] \)
Is the system time-invariant? (Yes) (No)  
(5 points)

Justify your answer:

\[
\begin{align*}
\gamma[2(n+n_0)] &\rightarrow n^2 x[2(n+n_0)] = y[n] \\
y[n+n_0] &\neq (n+n_0)^2 x[2(n+n_0)]
\end{align*}
\]

\[\therefore y[n+n_0] \neq y_1[n] \quad \therefore \text{Time-variant}.
\]

(c) \( y[n] = x^3[2n] \)
Is the system linear? (Yes) (No)  
(5 points)

Justify your answer:

\[
\begin{align*}
a x_1[n] &\rightarrow a y_1[n] = a x_1^3[2n] \\
b x_2[n] &\rightarrow b y_2[n] = b x_2^3[2n]
\end{align*}
\]

\[
x[n] = a x_1[n] + b x_2[n] \rightarrow y[n] = (a x_1[n] + b x_2[n])^3
\]

\[\neq a y_1[n] + b y_2[n]
\]

\[\therefore \text{It is NOT linear!}
\]