DIGITAL FILTER DESIGN

FIR Design Methods

1) Windowing
   a) Truncation
   b) General windowing

2) Frequency Sampling ~ will cover later in connection with the DFT

3) Computer-Aided Optimization
   a) Parks - McClellan (√) ~ widely used
   b) Linear Programming

1a) Truncation

Illustrate by example.

Suppose we want a LPF with frequency response:

We might choose the filter coefficients \( \{h_n\} \) to be

\[
d_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} D(\omega) e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{\omega_c}{\pi} \text{sinc} \omega_c n
\]

(Then frequency response \( H_d(\omega) = \text{DTFT} \left[ \{d_n\} \right] = D(\omega) \) exactly.)

Look at the \( d_n \):

\[
D(\omega)
\]

\(-\pi\) \(
\omega_c\) \(\pi\) \(\omega\)
This sequence is infinite in length and noncausal. So, let’s choose (assume N odd, for now):

\[
h_n = \begin{cases} 
  d_n - \frac{N-1}{2} & 0 \leq n \leq N - 1 \\
  0 & \text{else}
\end{cases}
\]

If N is large, then \( h_n \) consists of all the larger values of \( d_n \) and

\[
H_d(\omega) \approx D(\omega) e^{-j\omega \frac{N-1}{2}}.
\]

But this approximation is poor in the following sense. The frequency response will look like:

The ripples are due to the Gibbs’ phenomenon. Get tall ripples at sharp transitions. As N increases, ripples become narrower and more numerous, but the heights of the ripples nearest the discontinuity remain large. We will soon develop our own explanation for why this occurs. For now, it is worth noting that the DTFT
\[ \sum_{n=-\infty}^{\infty} d_n e^{-j\omega n} \]

is actually a Fourier series expansion of \( D(\omega) \). We had not thought of the DTFT in this way before, because there was no advantage in doing so. Here, though, we note that \( H_d(\omega) \) is obtained by truncating this Fourier series (i.e., choosing \( h_n = d_{n-N-1/2} \) for \( 0 \leq n \leq N - 1 \)).

Gibbs and other mathematicians studied truncation of Fourier series and showed that ripples will occur around locations where the periodic function is discontinuous. These ripples can be made narrower and to bunch up around the points of discontinuity by taking \( N \) larger. However, a larger \( N \) does not reduce the heights of the ripples!

How can we reduce the ripple heights?

**1b) General Windowing**

Given \( D(\omega) \) and the corresponding infinite-length \( \{d_n\} \), choose the coefficients \( \{h_n\}_{n=0}^{N-1} \) to be (again, assume \( N \) odd for now):

\[ h_n = w_n \cdot d_n, \quad 0 \leq n \leq N - 1 \]

where \( w_n = 0, \quad n \notin [0, N - 1] \), is a window sequence that gently tapers to zero.

For truncation we used

\[ w_n = \begin{cases} 1 & 0 \leq n \leq N - 1 \\ 0 & \text{else} \end{cases} \]

which does *not* taper gently to zero. Smoother windows can lead to much lower ripple.

An example of a good window is the Hamming window:

\[ w_n = .54 - .46 \cos \left( \frac{2\pi n}{N - 1} \right), \quad 0 \leq n \leq N - 1. \]

This is plotted below.
How does the window choice affect the frequency response of the designed filter \( \{h_n\}_{n=0}^{N-1} \)?

Let

\[ g_n = d_n \frac{N-1}{2} \cdot \]

Then:

\[ h_n = w_n g_n \]

desired u.p.r. after shifting

window coeffs. of designed FIR filter

In the frequency domain this corresponds to:

\[
H_d(\omega) = \sum_n w_n g_n e^{-j\omega n}
\]

\[
= \sum_n w_n \frac{1}{2\pi} \int_{-\pi}^{\pi} G_d(\theta) e^{j\theta n} d\theta \ e^{-j\omega n}
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} G_d(\theta) \sum_n w_n e^{-jn(\omega-\theta)} d\theta
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} G_d(\theta) \ W_d(\omega-\theta) d\theta \quad (\ast)
\]

Notice that this has the form of a convolution. Since the integrand is periodic and we integrate over only a single period, this is called a periodic convolution.
Since our goal is to have $H_d(\omega) \approx G_d(\omega)$, we see from (*) that we would like

$$W_d(\omega) = 2\pi \delta(\omega)$$

But, $W_d(\omega) = 2\pi \delta(\omega) \Rightarrow$

$$w_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\pi \delta(\omega) e^{j\omega n} d\omega = 1 \quad \text{for all } n$$

$\Rightarrow w_n$ is not a window sequence!

$w_n$ nonzero only on $n \in [0, N-1]$ results in (at best) either:

i) $W_d(\omega)$ is a narrow pulse around $\omega = 0$, but has high sidelobes.

or

ii) $W_d(\omega)$ is wider around the origin but the sidelobes are lower.

So, we have a tradeoff.

From (*) we see that the ripple in $H_d(\omega)$ is caused by integrating the product of $G_d$ and $W_d$ as the ripples in $W_d$ are shifted across the discontinuity of $G_d$. Likewise, the width of the transition band for $H_d$ will depend on the width of the mainlobe (center lobe) of $W_d$. We conclude:

High sidelobes of $W_d \Rightarrow$ Large ripple in $H_d$.

Wide center lobe of $W_d \Rightarrow$ Wide transition band in $H_d$.

Our goal is to achieve moderately low sidelobes and a moderately narrow transition width in $W_d$ simultaneously.

What does $W_d(\omega)$ look like for some common windows?

**Truncation**

$$|W_d(\omega)| = \left| \sum_{n=0}^{N-1} e^{-j\omega n} \right| = \left| \frac{\sin \frac{N}{2} \omega}{\sin \frac{1}{2} \omega} \right|$$
11.6

Plot on log scale:

\[ |W_d(\omega)| = \left| \frac{\sin \frac{N+1}{4} \omega}{\sin \frac{1}{2} \omega} \right|^2 \]

Plot on log scale:
From this plot, we see that the triangular window has a mainlobe that is twice as wide as that of the rectangular window, but the highest sidelobe is reduced by 12 dB.

**Hamming:**

\[ w_n = .54 - .46 \cos \left( \frac{2\pi n}{N - 1} \right) \quad 0 \leq n \leq N - 1 \]

Plot of \( |W_d(\omega)| \) on a log scale:

From this plot, we see that the Hamming window has essentially the same mainlobe width as the triangular window, and sidelobes that are reduced an additional 16 dB to \(-41\) dB. Thus, the
Hamming window is preferred over the triangular window. Comparing the Hamming window to the truncation window, we see that the highest sidelobe is reduced by 28 dB (more than a factor of 10) at the expense of increasing the mainlobe width by a factor of 2.

Best window: Kaiser

\[
\begin{equation}
    w_n = I_0 \left[ \beta \left( 1 - \left[ \left( \frac{n - \frac{N-1}{2}}{\frac{N-1}{2}} \right)^2 \right]^{1/2} \right) \right] \\
    0 \leq n \leq N - 1
\end{equation}
\]

\(I_0\) is the zeroth-order modified Bessel function of the first kind:

\[
   I_0(x) = \frac{1}{\pi} \int_0^\pi e^{\pm x \cos \theta} \, d\theta = \frac{1}{\pi} \int_0^\pi \cosh (x \cos \theta) \, d\theta
\]

The choice of \(\beta\) affects the tradeoff between the mainlobe width and sidelobe heights. \(\beta\) is user specified.

The Kaiser window can achieve slightly narrower mainlobe with the same sidelobe height as Hamming window.
Typical Frequency Responses Using Window Design

If the desired magnitude response is

$$|G_d(\omega)|$$

then the frequency response of a truncation design may look like:

$$|H^T_d(\omega)| = |G_d(\omega) \ast W^T_d(\omega)|$$

and the frequency response of a Hamming design will look like:

$$|H^H_d(\omega)| = |G_d(\omega) \ast W^H_d(\omega)|$$

So, Hamming widens the transition band by a factor of two, but greatly reduces ripple.
The actual filter design procedure is

\[ h_n = w_n g_n \quad 0 \leq n \leq N - 1. \]

The above Fourier-domain concepts are to help us visualize the resulting \( H_d(\omega) \).

Now, so far we have considered only the case with \( N \) odd, where we defined

\[ g_n = d_{n - \frac{N-1}{2}} \]

where \( D(\omega) \) is the desired \( H_d(\omega) \).

How do we find \( g_n \) if \( N \) is even? Answer: Select \( g_n \) as shown in the following procedure, which works for both \( N \) odd and \( N \) even.

**General Window Design Procedure**

To design a generalized linear phase \( \{h_n\}_{n=0}^{N-1} \) with \( |H_d(\omega)| \approx D(\omega) \) do this:

1) Let \( G_d(\omega) = D(\omega) e^{-j\frac{N-1}{2}\omega} \)

2) Find \( g_n = \text{DTFT}^{-1} [G_d(\omega)] \)

3) Let \( h_n = w_n g_n \).

**Notes:**

1) For \( N \) odd this procedure gives \( g_n = d_{n - \frac{N-1}{2}} \) as before. For \( N \) even, steps 1) and 2) give \( \{g_n\} \) as an interpolated set of values lying between \( \{d_{n - \frac{N}{2}}\} \) and \( \{d_{n - \frac{N-2}{2}}\} \).

2) We wish to know whether \( H_d(\omega) \), designed via the window method, will have generalized linear phase. The answer is ordinarily yes, since \( \{h_n\} \) will be symmetric or antisymmetric if \( \{w_n\} \) is symmetric and \( \{g_n\} \) is either symmetric or antisymmetric.

**FIR Window Design Examples**

**Example**

Design generalized linear-phase, low-pass FIR filters having coefficients \( \{h_n\}_{n=0}^{29} \) and cutoff \( \omega_c = \frac{\pi}{4} \) using the window design procedure with both truncation and Hamming windows.
Solution

\[ G_d(\omega) = D(\omega)e^{-j\frac{\omega}{2}} \]

\[ = \begin{cases} e^{-j\frac{29}{2}\omega} & |\omega| \leq \frac{\pi}{4} \\ 0 & \frac{\pi}{4} < |\omega| \leq \pi \end{cases} \]

\[ \Rightarrow g_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j\frac{29}{2}\omega} e^{j\omega n} d\omega \]

\[ = \frac{1}{2\pi} e^{j\frac{29}{2}\omega} \left| \begin{array}{c} \pi \\ \frac{\pi}{4} \end{array} \right| - \left| \begin{array}{c} \frac{\pi}{4} \\ -\frac{\pi}{4} \end{array} \right| \]

\[ = \frac{\sin \left( \frac{\pi}{4} \left( n - \frac{29}{2} \right) \right)}{\pi \left( n - \frac{29}{2} \right)} \]

\[ = \frac{1}{4} \text{sinc} \left( \frac{\pi}{4} \left( n - \frac{29}{2} \right) \right) \]

Now, for truncation \( h_n = w_n g_n \) with \( w_n = \begin{cases} 1 & 0 \leq n \leq 29 \\ 0 & \text{else} \end{cases} \).

So:

\[ h_n = \frac{1}{4} \text{sinc} \left( \frac{\pi}{4} \left( n - \frac{29}{2} \right) \right) \quad 0 \leq n \leq 29 \]
For the Hamming window design we have

\[
h_n = \left[ 0.54 - 0.46 \cos \frac{2\pi n}{29} \right] \frac{1}{4} \text{sinc} \frac{\pi}{4} \left( \frac{n - 29}{2} \right) \quad 0 \leq n \leq 29
\]

**Example**

Design generalized linear phase high-pass FIR filters having coefficients \( h_n \) \( n=0 \) and cutoff \( \omega_c = \frac{2\pi}{3} \) using the window design procedure with both truncation and Hamming windows.

**Solution**

\[
G_d(\omega) = \begin{cases} 
  e^{-j30\omega} & \frac{2\pi}{3} \leq |\omega| \leq \pi \\
  0 & |\omega| < \frac{2\pi}{3}
\end{cases}
\]

\[ \Rightarrow g_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} G_d(\omega) e^{j\omega n} d\omega \]

periodic with period = \( 2\pi \)

\[ = \frac{1}{2\pi} \int_{0}^{2\pi} G_d(\omega) e^{j\omega n} d\omega \quad (*) \]

\[ = \frac{1}{2\pi} \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} e^{-j30\omega} e^{j\omega n} d\omega \quad (\square) \]

\[ = \frac{e^{j\omega(n-30)}}{2\pi j(n-30)} \frac{4\pi}{3} \quad (\Delta) \]
\[
\begin{align*}
  &= \frac{1}{2\pi j(n-30)} \left[ e^{\frac{4\pi}{3}(n-30)} - e^{-\frac{2\pi}{3}(n-30)} \right] \\
  &= \frac{1}{2\pi j(n-30)} e^{j\pi(n-30)} \left[ e^{\frac{\pi}{3}(n-30)} - e^{-\frac{j\pi}{3}(n-30)} \right] \\
  &= \frac{1}{2\pi j(n-30)} (-1)^n 2 j \sin \frac{\pi}{3} (n-30) \\
  &= (-1)^n \frac{1}{3} \text{sinc} \frac{\pi}{3} (n-30)
\end{align*}
\]

So, using a truncation window gives:

\[
\boxed{h_n = (-1)^n \frac{1}{3} \text{sinc} \frac{\pi}{3} (n-30) \quad 0 \leq n \leq 60}
\]

and applying a Hamming window gives:

\[
\boxed{h_n = \left[ .54 - .46 \cos \frac{2\pi n}{60} \right] (-1)^n \frac{1}{3} \text{sinc} \frac{\pi}{3} (n-30) \quad 0 \leq n \leq 60}
\]

Notes:
1) Since \( G_d(\omega) \) is nonzero on two subintervals of \(-\pi \leq \omega \leq \pi\) for the high-pass case, it can save algebra if we use periodicity to rewrite the inverse DTFT as in (*) across the interval \( 0 \leq \omega \leq 2\pi \). This trick is straightforward if \( N \) is odd. For \( N \) even, however, there are two extra things to think about. First, in view of the table on p. 35.8, we should use an odd-symmetric design with Type 2 generalized linear phase. Second, for \( N \) even, the slope of the phase, \(-\frac{N-1}{2}\), is noninteger and the phase will take a jump of \( \pi \) at \( \omega = \pi \). Thus, \( G_d(\omega) \) seemingly will have two different forms on the interval \( 0 \leq \omega \leq 2\pi \). See the next example for details.

2) Since the denominator of (\( \Delta \)) is zero at \( n = 30 \), we cannot presume that (\( \Delta \)) follows from (\( \square \)) at \( n = 30 \). Thus, we must be careful to check that our expressions for \( \{h_n\} \) hold at \( n = 30 \). Our final expression for \( g_n \), which follows from (\( \Delta \)), gives:

\[
g_{30} = (-1)^{30} \frac{1}{3} \cdot 1 = \frac{1}{3}
\]

Evaluating (\( \square \)) at \( n = 30 \) gives
which agrees with (A). Thus, our expressions for $h_n$ are valid for $0 \leq n \leq 60$.

Now, let’s change $N$ in the previous example from $N = 61$ to $N = 62$ and see how the algebra associated with the design changes for $N$ even.

**Example**

Design a generalized linear phase high-pass FIR filter having coefficients $\{h_n\}_{n=0}^{61}$ and cutoff $\omega_c = \frac{2\pi}{3}$ using the window design procedure with a Hamming window.

**Solution**

A filter with real-valued unit-pulse response satisfies $G_d(\omega) = G_d^*(-\omega)$. Thus, for an antisymmetric design with Type 2 generalized linear phase, we have

$$G_d(\omega) = \begin{cases} 
e^{j\left(\frac{\pi}{2} + \frac{61}{2}\omega\right)} & 2\pi \leq \omega < \pi \\ 0 & |\omega| < \frac{2\pi}{3} \\ 
e^{j\left(-\frac{\pi}{2} + \frac{61}{2}\omega\right)} & -\pi < \omega \leq -\frac{2\pi}{3} \end{cases}$$

$$= \begin{cases} j \ne^{-\frac{61}{2}\omega} & 2\pi \leq \omega < \pi \\ 0 & |\omega| < \frac{2\pi}{3} \\ -j \ne^{\frac{61}{2}\omega} & -\pi < \omega \leq -\frac{2\pi}{3} \end{cases} \quad (*)$$
As in the last example we can write

\[ g_n = \frac{1}{2\pi} \int_{-\pi/3}^{\pi/3} G_d(\omega) e^{j\omega n} d\omega \quad (**) \]

Equation (*) specifies \( G_d(\omega) \) on \( |\omega| < \pi \). To find \( G_d(\omega) \) for \( \pi < \omega < \frac{4\pi}{3} \) consider

\[
G_d(\omega) = -j e^{-\frac{61}{2} \omega} \quad \text{in here}
\]

\[
G_d(\omega) = j e^{-\frac{61}{2} \omega} \quad \text{in here}
\]

\[
G_d(\omega) = G_d(\omega - 2\pi)
\]

\[
= -j e^{-\frac{61}{2} (\omega - 2\pi)}
\]

\[
= j e^{-\frac{61}{2} \omega}
\]

So, \( G_d(\omega) \) in (**) maintains the same form across the full range of integration in (**) and we have

\[ g_n = \frac{1}{2\pi} \int_{-\pi/3}^{\pi/3} \left[ j e^{-\frac{61}{2} \omega} e^{j\omega n} e^{j\omega} \right] d\omega \]

\[ = e^{j\omega(n - \frac{61}{2})} \left[ \frac{4\pi}{2\pi} \right]^{3/2} \]

\[ = \frac{1}{2\pi \left( n - \frac{61}{2} \right)} \left[ e^{-j\frac{4\pi}{3} \left( n - \frac{61}{2} \right)} - e^{-j\frac{2\pi}{3} \left( n - \frac{61}{2} \right)} \right] \]
\[
= \frac{1}{2\pi \left( \frac{n - 61}{2} \right)} e^{j\pi \left( \frac{n - 61}{2} \right)} \left[ e^{j\frac{\pi}{3} \left( \frac{n + 61}{2} \right)} - e^{-j\frac{\pi}{3} \left( \frac{n + 61}{2} \right)} \right]
\]
\[
= \frac{1}{2\pi \left( \frac{n - 61}{2} \right)} (-1)^n 2\sin \left[ \frac{\pi}{3} \left( n - \frac{61}{2} \right) \right]
\]
\[
= (-1)^n \frac{1}{3} \sin \frac{\pi}{3} \left( n - \frac{61}{2} \right)
\]

Applying the Hamming window gives

\[
h_n = \left[ .54 - .46 \cos \frac{2\pi n}{61} \right] (-1)^n \frac{1}{3} \sin \frac{\pi}{3} \left( n - \frac{61}{2} \right)
\]

0 \leq n \leq 61

Example

Window design of bandpass filter. Find \( h_n^{N-1} \) so that \( |H_d(\omega)| \) approximates

For lowpass filter had \( d_n = \frac{\omega_c}{\pi} \sin \omega_c n \). By modulation property, here for BP case we expect

\( d_n \approx (\cos \omega_o n) \frac{\omega_c}{\pi} \sin \left[ \omega_c n \right] \). Let’s see:

\[
d_n = \frac{1}{2\pi} \int_{-\omega_o - \omega_c}^{-\omega_o + \omega_c} 1 \ e^{j\omega n} \ d\omega + \frac{1}{2\pi} \int_{\omega_o - \omega_c}^{\omega_o + \omega_c} 1 \ e^{j\omega n} \ d\omega
\]
\[
= e^{j\omega n} \left|_{-\omega_o - \omega_c}^{\omega_o + \omega_c} \right. + e^{j\omega n} \left|_{\omega_o - \omega_c}^{\omega_o + \omega_c} \right.
\]
\[
= \frac{e^{j\omega n}}{2\pi jn} \left|_{-\omega_o - \omega_c}^{\omega_o + \omega_c} \right. + \frac{e^{j\omega n}}{2\pi jn} \left|_{\omega_o - \omega_c}^{\omega_o + \omega_c} \right.
\]
\[
\frac{e^{jn(\omega_o + \omega_c)} - e^{jn(\omega_o - \omega_c)} + e^{jn(\omega_o + \omega_c)} - e^{jn(\omega_o - \omega_c)}}{2\pi jn}
\]

\[
= \frac{1}{2\pi jn} \left[ e^{-jn\omega_o} \left( e^{jn\omega_c} - e^{-jn\omega_c} \right) + e^{jn\omega_o} \left( e^{jn\omega_c} - e^{-jn\omega_c} \right) \right]
\]

\[
= \frac{1}{\pi n} \left[ e^{-jn\omega_o} \sin \omega_c n + e^{jn\omega_o} \sin \omega_c n \right] = \frac{\sin \omega_c n}{\pi n} 2 \cos \omega_o n
\]

\[
= 2 \left( \cos \omega_o n \right) \frac{\omega_c}{\pi} \text{sinc} \omega_c n \quad \leftarrow \text{what we expected.}
\]

Now, to design \( h_n \) with linear phase, need to incorporate shift to give:

\[
g_n = 2\cos \omega_o \left( n - \frac{N - 1}{2} \right) \frac{\omega_c}{\pi} \text{sinc} \omega_c \left( n - \frac{N - 1}{2} \right) \quad 0 \leq n \leq N - 1
\]

Windowed coefficients are then

\[
h_n = w_n \cdot g_n \quad 0 \leq n \leq N - 1 \quad \text{coefficients for FIR filter where } w_n \text{ is a Hamming or other window.}
\]

In MATLAB, the command to design FIR filters using the window method is called \texttt{fir1}. 
Parks-McClellan

Parks and McClellan developed a computer program for solving the following problem:

Define the error

\[ E(\omega) = W(\omega)[D(\omega) - R(\omega)] \]

where \( D(\omega) = R(\omega)e^{j(\frac{\omega - \omega_0}{2})} \) if \( 0 \leq \omega_0 \leq \pi \) or \( \pi \leq \omega_0 \leq \frac{\pi}{2} \).

Let \( \omega_p, \omega_s \) be the passband and stopband cutoff frequencies so that \( D(\omega) \) might look like:

![Diagram of D(\omega)]

Ordinarily \( D(\omega) \) is unspecified on \( \omega_p < \omega < \omega_s \) because we are not concerned with the precise shape of \( H_d(\omega) \) or \( R(\omega) \) in this transition band.

The Parks-McClellan algorithm finds \( \{h_n\}_{n=0}^{N-1} \) that minimizes

\[ \max_{0 \leq \omega \leq \omega_p, \omega_s \leq \omega \leq \pi} |E(\omega)| \]

This error measure is called the minimax or Chebyshev error norm.

In the standard P-M algorithm, \( W(\omega) \) can be selected to have one value, \( W_p \), on the passband and another, \( W_s \), on the stopband. Frequently, \( W_s \) is chosen larger than \( W_p \) so that the designed filter will have a smaller stopband error than passband error.

The program user specifies:
The designed filter has equiripple behavior:

\[ H_d(\omega) \]

1.0

\[ 1 \pm \delta_p \]

\[ \omega_p \]

\[ \omega_s \]

\[ \delta_s \]

How are the ripple heights related to \( W_p \) and \( W_s \)?

Answer: They satisfy \( \delta_p W_p = \delta_s W_s \).

Thus, \( \frac{\delta_p}{\delta_s} = \frac{W_s}{W_p} \).

What filter order is required to meet given specifications?

Answer: It has been found empirically that

\[ N \approx -10 \log_{10} \left( \frac{\delta_p \delta_s}{2.324(\omega_s - \omega_p)} \right) - 13 \]

Note: The filter order is not too sensitive to \( \delta_p \) and \( \delta_s \). But, \( N \) is inversely proportional to the transition bandwidth! Halving the transition bandwidth doubles the required filter length!

All students will get experience with the Parks-McClellan algorithm in a Matlab assignment.

**Linear Programming**
Linear programming is a fairly general optimization algorithm that can solve the Parks-McClellan problem and many others. LP is slower, but its generality can be exploited to incorporate time-domain or additional frequency-domain constraints.

For example, it is possible to eliminate ripple in the passband by constraining the derivative of \( R(\omega) \) to be \( \leq 0 \) in the passband.

The result is a monotone decreasing response in the passband, with ripple remaining in the stopband:

\[
\begin{align*}
\omega_s & \quad \omega_p \\
H(\omega) & \leq 1.0
\end{align*}
\]

Incorporating the monotone passband constraint increases the required filter order slightly.

In MATLAB, the command to design FIR filters using the P-M method is called `remez`.

**Frequency Sampling FIR Filter Design**

Given \( G_d(\omega) \sim \) desired frequency response with linear phase

Idea: Choose \( H_d(\omega) \) to agree with \( G_d(\omega) \) at \( \omega = \frac{2\pi}{N} m, \ 0 \leq m \leq N - 1 \).

Take \( \{h_n\}_{n=0}^{N-1} \) to be DFT\(^{-1}\) \( \left\{ G_d\left( \frac{2\pi m}{N} \right) \right\}_{m=0}^{N-1} \), i.e.,

\[
h_n = \frac{1}{N} \sum_{m=0}^{N-1} G_d\left( \frac{2\pi m}{N} \right) e^{j \frac{2\pi}{N} nm} \quad 0 \leq n \leq N - 1
\]  

(FS)
Using \(\{h_n\}_{n=0}^{N-1}\) as our FIR filter coefficients will give a frequency response \(H_d(\omega)\) satisfying

\[
H_d(\omega) = G_d(\omega) \text{ at } \omega = \frac{2\pi m}{N} \quad 0 \leq m \leq N - 1
\]

Why?

\[
H_d(\omega) \bigg|_{\omega = \frac{2\pi m}{N}} = \text{DFT} \left[ \{h_n\} \right]
\]

\[
= \text{DFT} \left[ \text{DFT}^{-1} \left[ \{G_d\left(\frac{2\pi m}{N}\right)\} \right] \right]
\]

by (FS)

\[
= G_d \left( \frac{2\pi m}{N} \right)
\]

But in general \(H_d(\omega) \neq G_d(\omega)\) for \(\omega \neq \frac{2\pi}{N} m\).

This approach can be too simplistic.

Example

Let \(G_d(\omega)\) be an ideal LPF with linear phase and delay \(\frac{N-1}{2}\).
Using (FS), will produce an $H_d(\omega)$ that agrees with $G_d(\omega)$ at the points shown above. The resulting $H_d(\omega)$ might look like:

There is a modified frequency sampling design procedure that works better. In the modified procedure, one or more frequency samples in the filter’s transition band are left unconstrained as free variables. The values of these free variables are then selected via linear programming to minimize some measure of ripple. This results in a wider transition band, but lower ripple, as suggested in the figures below.
Note: If you take $G_d(\omega)$ to have zero phase, the designed $H_d$ will still pass through the correct samples of $G_d$, but the response will be terrible off the grid $\omega = \frac{2\pi m}{N}$. Hoping for zero phase (no delay) in a causal filter is a pipe dream!

**Example**

Use the frequency sampling method to design a linear-phase HPF $\{h_n\}_{n=0}^{60}$, with cutoff $\omega_c = .6\pi$.

**Solution**

$$N = 61 \Rightarrow \text{phase} = -\frac{N-1}{2} \omega = -30 \omega$$

$$\Rightarrow G_d(\omega) = \begin{cases} 
0 & 0 \leq \omega < .6\pi \\
e^{-j30\omega} & .6\pi \leq \omega \leq 1.4\pi \\
0 & 1.4\pi < \omega \leq 2\pi 
\end{cases}$$

$$\Rightarrow G_d\left(\frac{2\pi m}{61}\right) = \begin{cases} 
0 & 0 \leq m \leq 18 \\
e^{-j30\frac{2\pi}{61}m} & 19 \leq m \leq 42 \\
0 & 43 \leq m \leq 60 
\end{cases}$$

Now, $\{h_n\}$ is the inverse DFT of $\left\{G_d\left(\frac{2\pi m}{61}\right)\right\}$ as given by

$$h_n = \frac{1}{61} \sum_{m=0}^{60} G_d\left(\frac{2\pi m}{61}\right) e^{\frac{2\pi i}{61}nm}$$
\[
\frac{1}{61} \sum_{m=19}^{42} e^{-j\frac{30\pi}{61}m} e^{j\frac{2\pi}{61}nm} = \frac{1}{61} \sum_{m=19}^{42} e^{j\frac{2\pi}{61}(n-30)m}
\]

Effect of linear phase is to circularly shift the \{h_n\} since replacing \(n-30\) with \(<n - 30>_{61}\) does not change this equation.

\[
\frac{1}{61} \sum_{k=0}^{23} e^{j\frac{2\pi}{61}(n-30)(k+19)}
\]

\(k = m - 19\)

\[
= \frac{1}{61} e^{j\frac{2\pi}{61}(n-30)19} \frac{1-e^{j\frac{2\pi}{61}(n-30)24}}{1-e^{j\frac{2\pi}{61}(n-30)}}
\]

\[
= \frac{1}{61} e^{j\frac{2\pi}{61}(n-30)31} \frac{e^{-j\frac{2\pi}{61}(n-30)12} - e^{j\frac{2\pi}{61}(n-30)12}}{e^{-j\frac{\pi}{61}(n-30)} - e^{j\frac{\pi}{61}(n-30)}}
\]

\[
= \frac{1}{61} e^{j\pi(n-30)} \frac{-2j\sin\left(\frac{2\pi}{61}(n-30)12\right)}{-2j\sin\left(\frac{\pi}{61}(n-30)\right)}
\]

\[
= \frac{1}{61} (-1)^n \frac{\sin\left(\frac{24\pi}{61}(n-30)\right)}{\sin\left(\frac{\pi}{61}(n-30)\right)} \quad 0 \leq n \leq 60
\]

Sampled periodic sinc

Here, the factor \((-1)^n\) makes this a high-pass filter rather than a low-pass filter. Since this term multiplies a periodic sinc, which is similar to a sinc, we expect that the resulting frequency
response will not be too different from that obtained using the window design procedure with a truncation window.

Let’s next consider the design of a low-pass filter \( \{h_n\}_{n=0}^{N-1} \) where \( N \) is even. This example will be more complicated than the previous one in two respects. First, since \( N \) is even, the formula for \( G_d(\omega) \) will take two different forms on \( 0 \leq \omega < 2\pi \). Second, since this is a low-pass filter, there will be two separate bands where \( G_d(\omega) \) is nonzero on \( 0 \leq \omega < 2\pi \).

**Example**

Frequency sampling design of \( \{h_n\}_{n=0}^{21} \), linear-phase LPF with \( \omega_c = \frac{\pi}{2} \).

**Solution**

\[
N = 22 \Rightarrow \text{phase} = -\frac{N - 1}{2} \omega = -\frac{21}{2} \omega
\]

\[
\Rightarrow G_d(\omega) = \begin{cases} 
e^{-j\frac{21}{2}\omega} & |\omega| \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |\omega| \leq \pi \end{cases}
\]

But, for a frequency sampling design, we need samples of \( G_d(\omega) \) on \( \omega \in [0, 2\pi) \). For \( N \) even, with \( \frac{N - 1}{2} \) noninteger, we must be careful! Consider the phase of \( G_d(\omega) \). Here for clarity, we will not wrap the phase inside the interval \((-\pi, \pi]\) and will instead show its linear extension.
\[ G_d(\omega) = e^{-j\frac{2\omega}{2}} \text{ in here} \]

\[ G_d(\omega) = e^{-j\frac{2\omega}{2}(\omega-2\pi}) \]

\[ = e^{j\pi} e^{-j\frac{2\omega}{2}} \]

\[ = -e^{-j\frac{2\omega}{2}} \text{ in here} \]

Note: Minus sign doesn’t occur if N is odd.

So:

\[
G_d(\omega) = \begin{cases} 
  e^{-j\frac{2\omega}{2}} & 0 \leq \omega \leq \frac{\pi}{2} \\
  0 & \frac{\pi}{2} < \omega < \frac{3\pi}{2} \\
  -e^{-j\frac{2\omega}{2}} & \frac{3\pi}{2} \leq \omega \leq 2\pi 
\end{cases}
\]

\[
\Rightarrow G_d\left(\frac{2\pi m}{22}\right) = \begin{cases} 
  e^{-j\frac{2\pi m}{22}} & 0 \leq m \leq 5 \\
  0 & 6 \leq m \leq 16 \\
  -e^{-j\frac{2\pi m}{22}} & 17 \leq m \leq 21 
\end{cases}
\]
Thus,

\[ h_n = \frac{1}{22} \sum_{m=0}^{5} e^{-j \frac{2\pi mn}{22}} e^{j \frac{2\pi}{22} mn} + \frac{1}{22} \sum_{m=17}^{21} e^{-j \frac{2\pi mn}{22}} e^{j \frac{2\pi}{22} mn} \]

\[ = \frac{1}{22} \sum_{m=0}^{5} e^{j \frac{2\pi}{22} (n - \frac{21}{2}) m} - \frac{1}{22} \sum_{m=17}^{21} e^{j \frac{2\pi}{22} (n - \frac{21}{2}) m} \]

Making the change of variable \( \ell = m - 22 \), the second sum can be rewritten as

\[ \sum_{\ell=-5}^{-1} e^{j \frac{2\pi}{22} (n - \frac{21}{2}) (\ell + 22)} = \sum_{\ell=-5}^{-1} e^{j \frac{2\pi}{22} (n - \frac{21}{2})} e^{-j 2\pi \ell} = -1 \]

Thus,

\[ h_n = \frac{1}{22} \sum_{m=-5}^{5} e^{j \frac{2\pi}{22} (n - \frac{21}{2}) m} \]

\[ = \frac{1}{22} \sum_{k=0}^{10} e^{j \frac{2\pi}{22} (n - \frac{21}{2}) (k - 5)} \]

\[ k = m + 5 \]

\[ = \frac{1}{22} e^{j \frac{2\pi}{22} (n - \frac{21}{2}) (-5)} \frac{1 - e^{j \frac{2\pi}{22} (n - \frac{21}{2}) 1}}{1 - e^{j \frac{2\pi}{22} (n - \frac{21}{2}) 1}} \]

\[ = \frac{1}{22} e^{-j \frac{10\pi}{22} (n - \frac{21}{2})} \frac{e^{j \frac{2\pi}{22} (n - \frac{21}{2}) \frac{1}{2} \frac{1}{2}}}{e^{j \frac{2\pi}{22} (n - \frac{21}{2}) \frac{1}{2}}} \cdot \frac{-j 2 \sin \frac{11\pi}{22} (n - \frac{21}{2})}{-j 2 \sin \frac{\pi}{22} (n - \frac{21}{2})} \]

\[ = \frac{1}{22} \frac{\sin \frac{\pi}{22} (n - \frac{21}{2})}{\sin \frac{\pi}{22} (n - \frac{21}{2})} \quad 0 \leq n \leq 21 \]

Thus, \( h_n \) is a sampled, periodic sinc, which is similar to the sinc-type design that would have resulted from a window design using a truncation window.
In general, the formula for the coefficients of either an FIR LPF or HPF, designed using the frequency sampling technique, are given by

\[
\text{LPF: } h_n = \frac{1}{N} \frac{\sin \left( \gamma \frac{\pi}{N} \left( n - \frac{N - 1}{2} \right) \right)}{\sin \left( \frac{\pi}{N} \left( n - \frac{N - 1}{2} \right) \right)}
\]

\[
\text{HPF: } h_n = (-1)^n \frac{1}{N} \frac{\sin \left( \gamma \frac{\pi}{N} \left( n - \frac{N - 1}{2} \right) \right)}{\sin \left( \frac{\pi}{N} \left( n - \frac{N - 1}{2} \right) \right)}
\]

where \( \gamma \) is the number of samples \( G_d \left( \frac{2\pi m}{N} \right) \), \( 0 \leq m \leq N - 1 \), that are nonzero (number of samples in the passband).