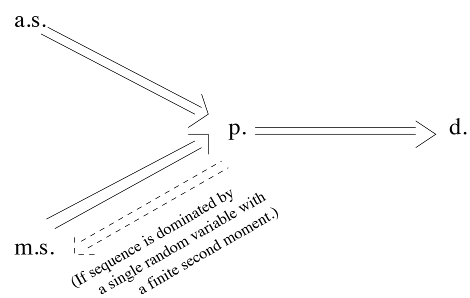


Law of large numbers Suppose $X_1, X_2, \ldots$ is a sequence of random variables such that each X_i has finite mean m. Let $S_n = X_1 + \cdots + X_n$. Then

(a) $\frac{S_n}{n} \overset{m.s.}{\to} m$ (hence also $\frac{S_n}{n} \overset{p.}{\to} m$ and $\frac{S_n}{n} \overset{d.}{\to} m$) if for some constant c, $\mathrm{Var}(X_i) \leq c$ for all i, and $\mathrm{Cov}(X_i, X_j) = 0$ $i \neq j$ (i.e. if the variances are bounded and the X_i's are uncorrelated).

(b) $\frac{S_n}{n} \overset{p.}{\to} m$ if $X_1, X_2, \ldots$ are *iid*. (weak law)

(c) $\frac{S_n}{n} \overset{a.s.}{\to} m$ if $X_1, X_2, \ldots$ are *iid*. (strong law)

Central limit theorem Suppose $X_1, X_2, \ldots$ are *i.i.d.*, each with mean μ and variance σ^2. Let $S_n = X_1 + \cdots + X_n$. Then $\frac{S_n - n\mu}{\sqrt{n}}$ converges in distribution to the $N(0, \sigma^2)$ distribution as $n \to \infty$.

Cramér's theorem Suppose $E[X_1]$ is finite, and that $E[X_1] < a$. Then for $\epsilon > 0$ there exists a number n_ϵ such that $P\left\{\frac{S_n}{n} \geq a\right\} \geq \exp(-n(l(a) + \epsilon))$ for all $n \geq n_\epsilon$. Combining this bound with the Chernoff inequality yields $\lim_{n\to\infty} \frac{1}{n} \ln P\left\{\frac{S_n}{n} \geq a\right\} = -l(a)$.

A **Brownian motion,** also called a *Wiener process*, with parameter σ^2 is a random process $W = (W_t : t \geq 0)$ such that

- $P\{W_0 = 0\} = 1$,
- W has independent increments,
- $W_t - W_s$ has the $N(0, \sigma^2(t - s))$ distribution for $t \geq s$, and
- W is sample path continuous with probability one.

A **Poisson process** with rate λ is a random process $N = (N_t : t \geq 0)$ such that

- N is a counting process,
- N has independent increments, and
- $N(t) - N(s)$ has the $Poi(\lambda(t - s))$ distribution for $t \geq s$.

Kolomorov forward equations for time-homogeneous, continuous-time, discrete-state Markov processes:

$$\frac{\partial \pi(t)}{\partial t} = \pi(t)Q \quad \text{or} \quad \frac{\partial \pi_j(t)}{\partial t} = \sum_{i \in \mathcal{S}} \pi_i(t) q_{ij} \quad \text{or} \quad \frac{\partial \pi_j(t)}{\partial t} = \sum_{i \in \mathcal{S}, i \neq j} \pi_i(t) q_{ij} - \sum_{i \in \mathcal{S}, i \neq j} \pi_j(t) q_{ji},$$

The orthogonality principle Let $\mathcal{V}$ be a closed, linear subspace of $L^2(\Omega, \mathcal{F}, P)$, and let $X \in L^2(\Omega, \mathcal{F}, P)$, for some probability space $(\Omega, \mathcal{F}, P)$. There exists a unique element Z^* (also denoted by $\Pi_{\mathcal{V}}(X)$) in $\mathcal{V}$ so that $E[(X - Z^*)^2 \leq E[(X - Z)^2]$ for all $Z \in \mathcal{V}$.

(a) (*Characterization*) Let W be a random variable. Then $W = Z^*$ if and only if the following two conditions hold:

(i) $W \in \mathcal{V}$

(ii) $(X - W) \perp Z$ for all Z in $\mathcal{V}$.

(b)(*Error expression*) $E[(X - Z^*)^2] = E[X^2] - E[(Z^*)^2]$.

Special cases:

$\mathcal{V} = \mathbb{R}$	$Z^* = E[X]$
$\mathcal{V} = \{g(Y) : \quad g : \mathbb{R}^m \to \mathbb{R},\ E[g(Y)^2] < +\infty\}$	$Z^* = E[X\|Y]$
$\mathcal{V} = \{c_0 + c_1 Y_1 + c_2 Y_2 + \ldots + c_n Y_n : c_0, c_1, \ldots, c_n \in \mathbb{R}\}$	$Z^* = \widehat{E}[X \mid Y]$

For random vectors: $\widehat{E}[X|Y] = E[X] + \mathrm{Cov}(X,Y)\mathrm{Cov}(Y,Y)^{-1}(Y - EY)$
and for $e = X - \widehat{E}[X|Y]:\quad \mathrm{Cov}(e) = \mathrm{Cov}(X) - \mathrm{Cov}(X,Y)\mathrm{Cov}(Y,Y)^{-1}\mathrm{Cov}(Y,X)$.

Conditional pdf for jointly Gaussian random vectors Let X and Y be jointly Gaussian vectors. Given $Y = y$, the conditional distribution of X is $N(\widehat{E}[X|Y = y], \mathrm{Cov}(e))$.

Maximum likelihood (ML) and maximum a posterior (MAP) estimators

$$\widehat{\theta}_{ML}(y) = \arg\max_{\theta} p_Y(y|\theta) \quad \widehat{\Theta}_{MAP}(y) = \arg\max_{\theta} p_{\Theta|y}(\theta|y) = \arg\max_{\theta} p_{\Theta}(\theta)p_Y(y|\theta) = \arg\max_{\theta} p_{\Theta,Y}(\theta, y)$$

Proposition Suppose X is irreducible, aperiodic, discrete-time discrete-state Markov

(a) All states are transient, or all are positive recurrent, or all are null recurrent.

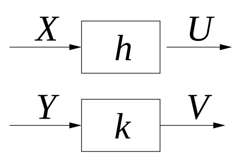
(b) For any initial distribution $\pi(0)$, $\lim_{t\to\infty} \pi_i(t) = 1/M_i$, with the understanding that the limit is zero if $M_i = +\infty$, where M_i is the mean time to return to state i starting from state i.

(c) An equilibrium probability distribution π exists if and only if all states are positive recurrent.

(d) If it exists, the equilibrium probability distribution π is given by $\pi_i = 1/M_i$. (In particular, if it exists, the equilibrium probability distribution is unique).

A **Karhunen-Loève (KL) expansion** for a random process $X = (X_t : a \leq t \leq b)$ is a representation of the form $X_t = \sum_{n=1}^{N} C_n \phi_n(t)$ with $N \leq \infty$ such that:
(1) the functions (ϕ_n) are orthonormal: $\langle \phi_m, \phi_n \rangle = I_{\{m=n\}}$, and
(2) the coordinate random variables C_n are mutually orthogonal: $E[C_m C_n^*] = 0$ if $n \neq m$.

Jointly WSS X and Y in LTI systems:



$R_{U,V} = h * \widetilde{k} * R_{XY} \leftrightarrow S_{U,V} = HK^* S_{X,Y}$