Problem set: VI's Introduction

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Exercise 1 *(Normal Cone)*

(a) Let $K \subset \mathbb{R}^n$ be a convex set and let $\bar{x} \in K$ be a given point. Show that the normal cone $\mathcal{N}_K(\bar{x})$ is closed convex cone.

(b) Let K_1 and K_2 be convex subsets in $\mathbb{R}^n$. Show the following relation

$$\mathcal{N}_{K_1}(x) + \mathcal{N}_{K_2}(x) \subset \mathcal{N}_{K_1 \cap K_2}(x) \qquad \text{for all } x \in K_1 \cap K_2.$$

(c) Show that the relation in (b) holds with equality if the sets K_1 and K_2 are such that $\mathrm{int}(K_1) \cap \mathrm{int}(K_2) \neq \emptyset$, where $\mathrm{int}(K)$ stands for the interior of a set K.

Exercise 2 *Let $X \subset \mathbb{R}^n$ be a closed convex set. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$ be convex and continuously differentiable functions. Consider the following optimization problem*

$$\begin{array}{ll} \textit{minimize} & f(x) \\ \textit{subject to} & g_i(x) \leq 0 \text{ for all } i = 1, \ldots, m, \\ & x \in X. \end{array} \tag{1}$$

Assume that the optimal value of the problem is finite. Also, assume that the Slater condition holds, i.e., there is $\bar{x} \in X$ such that $g_i(\bar{x}) < 0$ for all i. Show that x^* is a solution to the preceding minimization problem if and only if a pair (x^*, μ^*) solves $VI(K, F)$ where $K = X \times \mathbb{R}^m_+$ and

$$F(x, \mu) = \begin{bmatrix} \nabla f(x) + \sum_{i=1}^m \mu_i \nabla g_i(x) \\ -g_1(x) \\ \vdots \\ -g_m(x) \end{bmatrix}.$$

Exercise 3 *(Generalized Nash game) Consider a game of N players. Each player i has a convex and continuously differentiable cost function $f_i(x_i, x_{-i})$. A player i wants to minimize his cost with respect to his decision variable x_i over a closed convex decision set $K_i(x_i, x_{-i}) = \{x_i \mid g_i(x_i, x_{-i}) \leq 0\}$ where g_i is a convex and continuously differentiable function. Note that the players' decision sets are coupled!*

Assume that the Slater condition holds for the decision sets, i.e., there is $\bar{x}$ such that $g_i(\bar{x}) < 0$ for $i = 1, \dots, N$. Formulate the generalized Nash equilibrium problem as a Mixed Complementarity problem.

Exercise 4 *Let $K \subseteq \mathbb{R}^n$ be a convex and closed set, and let $F : K \to \mathbb{R}^n$ be a continuous mapping. Show that x^* is a solution to $VI(K, F)$ if and only if*

$$F^{\text{nat}}_{K,A}(x^*) = 0,$$

where

$$F^{\text{nat}}_{K,A}(x) \triangleq x - \Pi_{K,A}[x - A^{-1}F(x)],$$

with A being a symmetric and positive definite $n \times n$ matrix. (See page 85 of the volume I of the book by Facchinei and Pang).