Lecture 9

Monotone VIs/CPs

Properties of cones and some existence results

October 6, 2008
Outline

▶ Properties of cones

▶ Existence results for monotone CPs/VIs

▶ Polyhedrality of solution sets
Properties of cones

Motivation:

- Recall that in the earlier section, existence statements required that the range of $x^{\text{ref}}$ satisfied a certain interioricity property: namely if

\[ F(x^{\text{ref}}) \in \text{int}( (K_\infty)^* ), \]

then SOL($K.F$) is nonempty and compact.

- Essentially, this reduces to characterizing the interior of the dual cone

- Given an arbitrary closed convex cone, we provide results for a vector to be in the relative interior of $C^*$.
Introduction

Let $C$ be a closed convex cone in $\mathbb{R}^n$

- The set $C \cap (-C)$ is a linear subspace in $\mathbb{R}^n$ called the **lineality space** of $C$; This is the largest subspace contained in $C$ and is denoted by

$$\text{lin } C = C \cap (-C).$$

- For $S_1, S_2 \subseteq \mathbb{R}^n$, we have $(S_1 + S_2)^* = S_1^* \cap S_2^*$.

- If $C_1$ and $C_2$ are closed convex cones such that the sum $C_1^* + C_2^*$ is closed then we have $(C_1 \cap C_2)^* = C_1^* + C_2^*$.

- We may then show that the linear hull of $C^*$ denoted by $\text{lin } C^*$ satisfies

$$\text{lin } C^* = (C \cap (-C))^\perp,$$

where $A^\perp$ is the orthogonal complement of $A^*$.

\*The set $v^\perp$ is a subspace containing vectors that are orthogonal to $v$
Relative interior

**Definition 1** Let $S$ be a subset of $\mathbb{R}^n$. The *relative interior* of $S$ is the interior of $S$ considered as a subset of the affine hull $\text{Aff}(S)$ and is denoted by $ri(S)$. The formal definition is

$$ri(S) = \{ x \in S : \exists \epsilon > 0, (N_\epsilon \cap \text{aff}(S)) \subset S \}.$$ 

Consider $S = \{(x, y, 0) : 0 \leq x, y \leq 1 \} \subset \mathbb{R}^3$.

- Its interior $\text{int}(S) = \emptyset$.
- Its relative interior is nonempty since $ri(S) = \{(x, y, 0) : 0 < x, y < 1 \}$. This follows from noting that $\text{aff}(S) = \{(x, y, 0) : x, y \in \mathbb{R} \}$

Consider $S = \{(x, y, z) : 0 \leq x, y, z \leq 1 \} \subset \mathbb{R}^3$. 

Game theory: Models, Algorithms and Applications 4
Its interior and relative interior are the same and are given by \( \text{int}(S) = \text{ri}(S) = \{(x, y, z) : 0 < x, y, z < 1\} \).

If \( S = \text{ri}(S) \), \( S \) is said to be relatively open

The relative boundary, \( rbd(S) = \overline{S} - \text{ri}(S) \).

**Prop:** Let \( C \) be a closed convex cone in \( \mathbb{R}^n \). Then the following are equivalent:

(a) \( v \in \text{ri}(C^*) \)

(b) For all \( x \neq 0, x \in C \cap \text{lin} C^*, v^T x > 0 \)

(c) For every scalar \( \eta > 0 \), the set \( S(v, \eta) \equiv \{x \in C \cap \text{lin} C^* : v^T x \leq \eta\} \) is bounded

(d) \( v \in C^* \) and the intersection \( C \cap v^\perp \) is a linear subspace

If any of the above holds, then \( C \cap v^\perp \) is equal to the lineality space of \( C \).
Solid and pointed cones

**Definition 2** A cone $C$ is pointed if $C \cap (-C) = \{0\}$. A set $S$ is solid if $\text{int}(S) \neq \emptyset$.

- $\mathbb{R}^n_+$ is pointed and solid
- $\text{pos}(A)$ is also pointed and solid

**Lemma 1** Let $C$ be a closed convex cone in $\mathbb{R}^n$. Then the following hold:

1. If $C$ is solid, then its dual $C^*$ is pointed.

2. If $C$ is pointed, then its dual $C^*$ is solid.

Thus $C$ is solid(pointed) if and only if $C^*$ is pointed(solid).

**Proof:**
1. If $C$ is solid, then to show that $C^*$ is pointed, we need to show that $C^* \cap (-C^*) = \{0\}$. Let $d \in (C^* \cap (-C^*))$. Therefore, we have $d^T x = 0$ for all $x \in C$. Since $C$ is closed and convex, we have $C = C^{**}$. Moreover, from $C$ being solid, we have $\text{int}(C) \neq \emptyset$. Let $y$ be an interior point of $C = C^{**}$. Then, by definition $y^T d > 0$ if $d \neq 0$. But this contradicts $d^T x = 0, \forall x \in C$. Therefore, $d = 0$ and $C^*$ is pointed.

2. Let $C$ be pointed. To show that $C^*$ has a nonempty interior, let $k$ be the maximum number of linearly independent vectors in $C^*$ and let $\{y^1, \ldots, y^k\}$ represent a collection of such vectors:

(a) If $k = n$, then $C^*$ contains the simplicial (pos) cone generated by these vectors and is nonempty. Since $C^*$ contains this cone, the result follows.

(b) If $k < n$, then the system $p^T y^i = 0, i = 1, \ldots, k$ for some $0 \neq p \in \mathbb{R}^n$. But these $k$ vectors span the space in $C^*$ implying that
\[ p^T y = 0, \forall y \in C^*. \] Consequently, \( p \in C^{**} \cap (-C^{**}) \) or \( p \in C \cap (-C') \) since \( C = C^{**} \). Therefore \( C \cap (-C') \neq \{0\} \), contradicting the pointedness of \( C \).

(c) To establish the last assertion, assume that \( C^* \) is pointed. Then by (b), we have that \( C^{**} \) is solid. But since \( C \) is closed and convex, then \( C = C^{**} \). Thus \( C \) is solid. \( C^* \) is solid implies \( C \) is pointed in a similar fashion.
Existence results

**Theorem 1** Let $K$ be a closed convex cone in $\mathbb{R}^n$ and let $F$ be a pseudo-monotone continuous map from $K$ into $\mathbb{R}^n$. Then the following three statements are equivalent:

(a) The CP($K,F$) is strictly feasible.

(b) The dual cone $K^*$ has a nonempty interior and SOL($K,F$) is a nonempty compact set.

(c) The dual cone $K^*$ has a nonempty interior and

$$K \cap [-(F(K)^*)] = \{0\}.$$ 

**Proof:**
(a) $\implies$ (b): If the CP$(K,F)$ is strictly feasible, then clearly $\text{int}\ K^*$ is nonempty. From theorem 2.3.5, we have that SOL$(K,F)$ is nonempty and compact.

**Theorem 2** Suppose $F$ is pseudo-monotone on $K$. If there exists an $x^{\text{ref}} \in K$ satisfying $F(x^{\text{ref}}) \in (\text{int}(K_\infty)^*)$, then SOL$(K,F)$ is nonempty, convex and compact.

(\textbf{b)} $\implies$ (\textbf{c)}: This follows from the next result (provided without proof) and by noting that $K = K_\infty$:

**Theorem 3** Let $K$ be a closed and convex set and $F$ be a pseudo-monotone continuous mapping. Then SOL$(K,F)$ is nonempty and bounded if and only if

$$K_\infty \cap [-F(K)^*)] = \{0\}.$$
(c) \implies (a): By prop 2.3.17, \( \deg(\mathbf{F}_{K}^{\text{nat}}, \Omega) = 1 \) for every bounded open set \( \Omega \) containing SOL(K,F). Let \( q \) be an arbitrary vector in \( \text{int} \ K^* \). For a fixed but arbitrary scalar \( \epsilon > 0 \), we have \( (\mathbf{F}_{K}^{\epsilon,\text{nat}}(x) = x - \Pi_{K}(x - F(x) + \epsilon q) \) be the natural map of the perturbed CP:

\[
K \ni x \perp F(x) - \epsilon q \in K^*.
\]

For sufficiently small \( \epsilon \) we have

\[
\deg(\mathbf{F}_{K}^{\epsilon,\text{nat}}, \Omega) = \deg(\mathbf{F}_{K}^{\text{nat}}, \Omega) = 1.
\]

Thus the perturbed CP has a solution, say \( x \) (from degree-theoretic results proved earlier). Since \( q \in \text{int} K^* \), \( x \) must belong to \( K \) and \( F(x) \) to \( K^* \). Hence, the CP(K,F) is strictly feasible.
**Special case: an affine map**

**Corollary 4** Let $K$ be a closed convex cone in $\mathbb{R}^n$ and let $F(x) = q + Mx$ be an affine pseudo-monotone map from $K$ into $\mathbb{R}^n$. Then the following three statements are equivalent:

1. There exists a vector $x \in K$ such that $Mx + q$ is in $K^*$.

2. The dual cone $K^*$ has a nonempty interior and SOL($K,F$) is a nonempty compact set.

3. The dual cone $K^*$ has a nonempty interior and

   $$[d \in K, M^T d \in (-K)^*, q^T d \leq 0] \implies d = 0.$$
It suffices to show that

\[ K \cap [-F(K)^*)] = \{0\} \]

is equivalent to the assertion in (c) above. In fact the former may be stated as

\[
[d \in K, M^T d \in (-K)^*, q^T d \leq 0] \\
= [d \in K, (q)^T d \leq 0, (Mx)^T d \leq 0 \forall x \in K] \\
= [d \in K, d^T (q + Mx) \leq 0, \forall x \in K] \\
= K \cap (-F(K))^*, F(K) = Mx + q, \\
= \{0\} \implies d = 0,
\]

giving us the required result.
Feasibility $\iff$ Solvability? - No!!

**Example 1** Consider the NCP($F$):

\[
F(x) \equiv \begin{pmatrix} 2x_1x_2 - 2x_2 + 1 \\ -x_1^2 + 2x_1 - 1 \end{pmatrix}, \quad (x_1, x_2) \in \mathbb{R}^2.
\]

We have

\[
\nabla F = \begin{pmatrix} 2x_2 & 2x_1 - 2 \\ -2 - 2x_1 & 0 \end{pmatrix} \preceq 0
\]

implying that $F$ is monotone on $\mathbb{R}^2_+$. The feasible region

\[
FEA(F) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 1, x_2 \geq 0\}.
\]

However, this NCP has no solution.
However it does have $\epsilon$-exact solutions. For a scalar $\epsilon \in (0, 1)$, the vector $x^\epsilon = (1 - \epsilon, 1/(2\epsilon))$. It may be verified that

$$\lim_{\epsilon \to 0} \min(x^\epsilon, F(x^\epsilon)) = 0$$

and there exists an $\bar{x}$ satisfying

$$\|\min(\bar{x}, F(\bar{x}))\| \leq \epsilon$$

for every $\epsilon > 0$. Such an $\bar{x}$ is called an $\epsilon$-approximate solution of $\text{NCP}(F)$.

This leads to two important questions:

- When does feasibility of $\text{CP}(K,F)$ imply its solvability?

  - $K$ is polyhedral and $F$ is affine
• F is strictly monotone on K and K is pointed but non-polyhedral

▶ Does every feasible monotone CP have $\varepsilon-$solutions? - yes!

**Theorem 5** Let $K$ be $\mathbb{R}^n_+$ and $F$ be a monotone affine map from $\mathbb{R}^n$ onto itself. Then the $CP(K,F)$ is solvable if and only if it is feasible.

**Proof:** Since $K$ is a polyhedral cone, we have that $K = K^{**}$. Moreover, let $F(x) = q + Mx$, where $M \succeq 0$. Consider the gap program given by

$$\min x^T(Mx + q)$$

subject to $x \in K$

$$Mx + q \in K^*.$$ 

This is a convex QP with an objective that is bounded below by zero on $K$. We then use the Frank-Wolfe theorem - which states that a quadratic
function, bounded below on a polyhedral set, attains its minimum on that set. Therefore the dual gap program attains a minimum at $x^*$. By the necessary conditions of optimality, there exists a $\lambda \in \mathbb{R}^n$ such that

$$K \ni x^* \perp v \equiv q + (M + M^T)x^* - M^T\lambda \in K^*$$

$$K^* \ni q + Mx^* \perp \lambda \in K^{**} = K.$$ 

Since $(x^*)^Tv = 0$ and $\lambda^Tv \geq 0$, we have

$$0 \geq (x^* - \lambda)^Tv$$

$$= (x^* - \lambda)^T(q + (M + M^T)x^* - M^T\lambda)$$

$$= (x^* - \lambda)^T(q + Mx^*) + (x^* - \lambda)^TM^T(x^* - \lambda)$$

$$\geq (x^* - \lambda)^T(q + Mx^*)$$

$$= (x^*)^T(Mx^* + q).$$
But \((x^*)^T(Mx^* + q) \geq 0\) implying that \((x^*)^T(Mx^* + q) = 0\) and \(x^*\) lies in SOL(K,F).
General strictly monotone $F$

**Theorem 6 (Th. 2.4.8)** Let $K$ be a pointed closed and convex cone in $\mathbb{R}^n$ and $F : K \to \mathbb{R}^n$ be a continuous map. Consider the following statements

(a) $F$ is strictly monotone on $K$ and $\text{FEA}(K,F)$ is nonempty;

(b) $F$ is strictly monotone on $K$ and $\text{CP}(K,F)$ is strictly feasible;

(c) the $\text{CP}(K,F)$ has a unique solution.

It holds that $(a) \iff (b) \implies (c)$.

**Proof:**

- Clearly $(b) \implies (a)$

- $(a) \implies (b)$ (omitted) - see Theorem 2.4.8 in FP-I

- Converse is proved in Th. 2.4.8

- $(b) \implies (c)$ follows from Th 2.3.5
F is affine monotone

Under the assumptions of affineness and monotonicity of F, we obtain the following:

**Lemma 2** Let $K$ be a convex cone in $\mathbb{R}^n$ and $F(x) = q + Mx$ where $M \succeq 0$. For any $y \in SOL(K, q, M)$ it holds that

$$SOL(K, q, M) = \{ x \in K : q + Mx \in K^*, (M^T + M)(x-y) = 0, q^T(x-y) = 0 \}$$

**Proof:**

- By proposition 2.3.6, we have from monotonicity that $x, y \in SOL(K, q, M)$:

  $$ (x - y)^T M (x - y) = 0. $$
For $x \neq y$, we have that $(M + M^T)(x - y) = 0$ which may be further simplified as

$$x^T(M + M^T)(x - y) = 0$$
$$x^T(M + M^T)x = x^T(M + M^T)y$$
$$x^TMx = \frac{1}{2}x^T(M + M^T)y$$
similarly $y^TMy = \frac{1}{2}x^T(M + M^T)y$

Since $x \in SOL(K, F)$, we have that $x^T(q + Mx) = 0 \iff x^Tq = -x^TMx$.

Similarly, $y^T(q + My) = 0 \iff y^Tq = -y^TMy$. But

$$y^TMy = x^TMx \implies x^Tq = -x^TMx = -y^TMy = y^Tq.$$
Therefore $x$ lies in the set on the right hand side of the specified set equation.

To prove the reverse inclusion, we need to show that if $x$ lies in

$$\{x \in K : q + Mx \in K^*, (M^T + M)(x - y) = 0, q^T(x - y) = 0\}$$

then $x^T(Mx + q) = 0$. Since $q^T(x - y) = 0$, $(M + M^T)(x - y) = 0$, we have

$$q^Tx = q^Ty, (M + M^T)x = (M + M^T)y.$$ 

The latter expression implies that $x^TMx = y^TMy$ implying that

$$x(Mx + q) = y^T(My + y) = 0.$$

The result follows.
**Implication:** The aforementioned lemma implies that $q^T x$ is a constant scalar and $(M + M^T)x$ is a constant vector for all $x \in SOL(K, q, M)$. Therefore if $K$ is polyhedral, then the set given by SOL(K,F) is also polyhedral.
Polyhedrality of Monotone AVI

Consider the **affine variational inequality** denoted by $AVI(K,q,M)$ which requires a vector $x$ such that

$$(y - x)^T(Mx + q) \geq 0, \quad \forall y \in K,$$

where $K$ is a polyhedral set. Note that if $F$ is non necessarily affine, we have a **linearly constrained VI**.

Before proving a result pertaining to the polyhedrality of the $SOL(K,q,M)$, we provide a result that gives a KKT characterization of the VI.

**Proposition 1** *Let $K$ be given by*

$$K \equiv \{ x \in \mathbb{R}^n : Ax \leq b, Cx = d \}$$
for some matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{l \times n}$. A vector $x$ solves the VI($K,F$) if and only if there exists $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^l$ such that

$$F(x) + C^T \mu + A^T \lambda = 0$$
$$Cx - d = 0$$
$$0 \leq b - Ax \perp \lambda \geq 0.$$ 

Proof:

- For a fixed $x$ and $K$ polyhedral, we may formulate the following LP:

$$\min \quad y^T F(x)$$
subject to $y \in K$, 

by noting that the VI is equivalent to finding an $x$ such that

$$y^T F(x) \geq x^T F(x) \quad \forall y \in K.$$ 

Therefore if $x \in SOL(K, F)$, then $x$ is a solution to this LP. But by LP duality, we have the existence of $(\lambda, \mu)$ such that the following holds:

$$F(x) + C^T \mu + A^T \lambda = 0$$
$$Cx - d = 0$$
$$0 \leq b - Ax \perp \lambda \geq 0.$$ 

Reverse holds in a similar fashion. 

We may now prove the polyhedrality of the solution set of $AVI(K,q,M)$. 


Clearly, when $K$ is a cone, this follows from the result for monotone $\text{CP}(K,q,M)$.

**Theorem 7** Let $K$ be polyhedral and $F(x) \equiv Mx + q$. The solution set of $\text{AVI}(K,q,M)$ is polyhedral.

**Proof:** Let $K$ be given as

$$K \equiv \{ x \in \mathbb{R}^n : Ax \leq b, Cx = d \}$$

for some matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{l \times n}$. However, $x$ is a solution to $\text{AVI}(K,q,m)$ if and only if $(x, \mu, \lambda)$ is a solution to

$$Mx + q + C^T\mu + A^T\lambda = 0$$

$$Cx - d = 0$$

$$0 \leq b - Ax \perp \lambda \geq 0.$$
But this is an mixed LCP (a special case of a monotone affine CP) and has a polyhedral solution set. However, the solution set of AVI(K,q,M) is a projection of this solution set given by the mapping:

\[
\begin{pmatrix}
    x \\
    \mu \\
    \lambda
\end{pmatrix}
\in \mathbb{R}^{n+\ell+m} \rightarrow x \in \mathbb{R}^n.
\]

Therefore the solution set of AVI(K,q,M) is also polyhedral. 
\[\blacksquare\]