Lecture 9

# Monotone VIs/CPs

## Properties of cones and some existence results

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# Outline

- Properties of cones
- Existence results for monotone CPs/VIs
- Polyhedrality of solution sets

# Properties of cones

## Motivation:

- Recall that in the earlier section, existence statements required that the range of $x^{\rm ref}$ satisfied a certain interioricity property: namely if

$$F(x^{\rm ref}) \in int((K_\infty)^*),$$

then SOL$(K.F)$ is nonempty and compact.

- Essentially, this reduces to characterizing the interior of the dual cone
- Given an arbitrary closed convex cone, we provide results for a vector to be in the relative interior of $C^*$.

## Introduction

Let $C$ be a closed convex cone in $\mathbb{R}^n$

- The set $C \cap (-C)$ is a linear subspace in $\mathbb{R}^n$ called the **lineality space** of C; This is the largest subspace contained in $C$ and is denoted by

$$\text{lin } C = C \cap (-C).$$

- For $S_1, S_2 \subseteq \mathbb{R}^n$, we have $(S_1 + S_2)^* = S_1^* \cap S_2^*$.
- If $C_1$ and $C_2$ are closed convex cones such that the sum $C_1^* + C_2^*$ is closed then we have $(C_1 \cap C_2)^* = C_1^* + C_2^*$.
- We may then show that the linear hull of $C^*$ denoted by lin$C^*$ satisfies $\text{lin}C^* = (C \cap (-C))^{\perp}$, where $A^{\perp}$ is the orthogonal complement of $A^*$.

*The set $v^{\perp}$ is a subspace containing vectors that are orthogonal to $v$

# Relative interior

**Definition 1** *Let $S$ be a subset of $\mathbb{R}^n$. The* ***relative interior*** *of $S$ is the interior of $S$ considered as a subset of the affine hull Aff(S) and is denoted by ri(S). The formal definition is*

$$ri(S) = \{x \in S : \exists \epsilon > 0, (\mathcal{N}_\epsilon \cap aff(S)) \subset S\}.$$

- Consider S $= \{(x, y, 0) : 0 \leq x, y \leq 1\} \subset \mathbb{R}^3$.
  - Its interior $int(S) = \emptyset$.
  - Its relative interior is nonempty since $\mathrm{ri}(S) = \{(x, y, 0) : 0 < x, y < 1\}$. This follows from noting that $aff(S) = \{(x, y, 0) : x, y \in \mathbb{R}\}$
- Consider S $= \{(x, y, z) : 0 \leq x, y, z \leq 1\} \subset \mathbb{R}^3$.

- Its interior and relative interior are the same and are given by $int(S) = \mathrm{ri}(S) = \{(x, y, z) : 0 < x, y, z < 1\}$.

- If S $=$ ri(S), S is said to be **relatively open**

- The relative boundary, $rbd(S) = \bar{S} - ri(S)$.

**Prop:** Let $C$ be a closed convex cone in $\mathbb{R}^n$. Then the following are equivalent:

(a) $v \in \mathrm{ri}(C^*)$

(b) For all $x \neq 0$, $x \in C \cap \mathrm{lin}\ C^*$, $v^T x > 0$

(c) For every scalar $\eta > 0$, the set $S(v, \eta) \equiv \{x \in C \cap \mathrm{lin}\ C^* : v^T x \leq \eta\}$ is bounded

(d) $v \in C^*$ and the intersection $C \cap v^\perp$ is a linear subspace

If any of the above holds, then $C \cap v^\perp$ is equal to the lineality space of $C$.

# Solid and pointed cones

**Definition 2** *A cone $C$ is pointed if $C \cap (-C) = \{0\}$. A set $S$ is solid if $int\ (S) \neq \emptyset$.*

- $\mathbb{R}^n_+$ is pointed and solid
- pos(A) is also pointed and solid

**Lemma 1** Let C be a closed convex cone in $\mathbb{R}^n$. Then the following hold:

1. If C is solid, then its dual $C^*$ is pointed.
2. If C is pointed, then its dual $C^*$ is solid.

Thus C is solid(pointed) if and only if $C^*$ is pointed(solid).

**Proof:**

**1.** If $C$ is solid, then to show that $C^*$ is pointed, we need to show that $C^* \cap (-C^*)$ is $\{0\}$. Let $d \in (C^* \cap (-C^*))$. Therefore, we have $d^T x = 0$ for all $x \in C$. Since $C$ is closed and convex, we have $C = C^{**}$. Moreover, from $C$ being solid, we have $\text{int}\,(C) \neq \emptyset$. Let $y$ be an interior point of $C = C^{**}$. Then, by definition $y^T d > 0$ if $d \neq 0$. But this contradicts $d^T x = 0, \forall x \in C$. Therefore, $d = 0$ and $C^*$ is pointed.

**2.** Let $C$ be pointed. To show that $C^*$ has a nonempty interior, let $k$ be the maximum number of linearly independent vectors in $C^*$ and let $\{y^1, \ldots, y^k\}$ represent a collection of such vectors:

(a) If $k = n$, then $C^*$ contains the simplicial (pos) cone generated by these vectors and is nonempty. Since $C^*$ contains this cone, the result follows.

(b) If $k < n$, then the system $p^T y^i = 0, i = 1, \ldots, k$ for some $0 \neq p \in \mathbb{R}^n$. But these $k$ vectors span the space in $C^*$ implying that

$p^T y = 0, \forall y \in C^*$. Consequently, $p \in C^{**} \cap (-C^{**})$ or $p \in C \cap (-C)$ since $C = C^{**}$. Therefore $C \cap (-C) \neq \{0\}$, contradicting the pointedness of C.

(c) To establish the last assertion, assume that $C^*$ is pointed. Then by (b), we have that $C^{**}$ is solid. But since C is closed and convex, then $C = C^{**}$. Thus C is solid. $C^*$ is solid implies $C$ is pointed in a similar fashion.

# Existence results

**Theorem 1** Let $K$ be a closed convex cone in $\mathbb{R}^n$ and let $F$ be a pseudo-monotone continuous map from $K$ into $\mathbb{R}^n$. Then the following three statements are equivalent:

(a) The CP(K,F) is strictly feasible.

(b) The dual cone K* has a nonempty interior and SOL(K,F) is a nonempty compact set.

(c) The dual cone K* has a nonempty interior and

$$K \cap [-(F(K)^*)] = \{0\}.$$

**Proof:**

- (a) $\Longrightarrow$ (b): If the CP(K,F) is strictly feasible, then clearly $\text{int}\, K^*$ is nonempty. From theorem 2.3.5, we have that SOL(K,F) is nonempty and compact.

  **Theorem 2** Suppose $F$ is pseudo-monotone on $K$. If there exists an $x^{\text{ref}} \in K$ satisfying $F(x^{\text{ref}}) \in (\text{int}(K_\infty)^*)$, then SOL$(K, F)$ is nonempty, convex and compact.

- (b) $\Longrightarrow$ (c): This follows from the next result (provided without proof) and by noting that $K = K_\infty$:

  **Theorem 3** *Let $K$ be a closed and convex set and $F$ be a pseudo-monotone continuous mapping. Then SOL(K,F) is nonempty and bounded if and only if*

$$K_\infty \cap [-(F(K)^*)] = \{0\}.$$

- (c) $\Longrightarrow$ (a): By prop 2.3.17, deg$(\mathbf{F}_K^{\rm nat}, \Omega) = 1$ for every bounded open set $\Omega$ containing SOL(K,F). Let $q$ be an arbitrary vector in $\mathrm{int}\ K^*$. For a fixed but arbitrary scalar $\epsilon > 0$, we have $(\mathbf{F}_K^{\epsilon,\rm nat}(x) = x - \Pi_K(x - F(x) + \epsilon q)$ be the natural map of the perturbed CP:

$$K \ni x \perp F(x) - \epsilon q \in K^*.$$

For sufficiently small $\epsilon$ we have

$$deg(\mathbf{F}_K^{\epsilon,\rm nat}, \Omega) = deg(\mathbf{F}_K^{\rm nat}, \Omega) = 1.$$

Thus the perturbed CP has a solution, say $x$ (from degree-theoretic results proved earlier). Since $q \in intK^*$, $x$ must belong to K and $F(x)$ to $K^*$. Hence, the CP(K,F) is strictly feasible.

# Special case: an affine map

**Corollary 4** Let $K$ be a closed convex cone in $\mathbb{R}^n$ and let $F(x) = q + Mx$ be an affine pseudo-monotone map from $K$ into $\mathbb{R}^n$. Then the following three statements are equivalent:

**1.** There exists a vector $x \in K$ such that $Mx + q$ is in $K^*$.

**2.** The dual cone K* has a nonempty interior and SOL(K,F) is a nonempty compact set.

**3.** The dual cone K* has a nonempty interior and

$$[d \in K, M^T d \in (-K)^*, q^T d \leq 0] \implies d = 0.$$

It suffices to show that

$$K \cap [-(F(K)^*)] = \{0\}$$

is equivalent to the assertion in (c) above. In fact the former may be stated as

$$\begin{aligned}
&[d \in K, M^T d \in (-K)^*, q^T d \leq 0] \\
=&[d \in K, (q)^T d \leq 0, (Mx)^T d \leq 0 \forall x \in K] \\
=&[d \in K, d^T(q + Mx) \leq 0, \forall x \in K] \\
=&K \cap (-F(K))^*, F(K) = Mx + q, \\
=&\{0\} \implies d = 0,
\end{aligned}$$

giving us the required result.

## Feasibility $\Longrightarrow$ Solvability? - No!!

**Example 1** *Consider the NCP(F):*

$$F(x) \equiv \begin{pmatrix} 2x_1x_2 - 2x_2 + 1 \\ -x_1^2 + 2x_1 - 1 \end{pmatrix}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

*We have*

$$\nabla F = \begin{pmatrix} 2x_2 & 2x_1 - 2 \\ 2 - 2x_1 & 0 \end{pmatrix} \succeq 0$$

*implying that $F$ is monotone on $\mathbb{R}^2_+$. The feasible region*

$$FEA(F) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 1, x_2 \geq 0\}.$$

*However, this NCP has no solution.*

*However it does have $\epsilon-$exact solutions. For a scalar $\epsilon \in (0,1)$, the vector $x^\epsilon = (1-\epsilon, 1/(2\epsilon))$. It may be verified that*

$$\lim_{\epsilon \to 0} \min(x^\epsilon, F(x^\epsilon)) = 0$$

*and there exists an $\bar{x}$ satisfying*

$$\|\min(\bar{x}, F(\bar{x}))\| \leq \epsilon$$

*for every $\epsilon > 0$. Such an $\bar{x}$ is called an $\epsilon-$ approximate solution of NCP(F).*

This leads to two important questions:

- When does feasibility of CP(K,F) imply its solvability?
  - K is polyhedral and F is affine

- F is strictly monotone on K and K is pointed but non-polyhedral

▶ Does every feasible monotone CP have $\epsilon-$solutions? - yes!

**Theorem 5** *Let $K$ be $\mathbb{R}^n_+$ and $F$ be a monotone affine map from $\mathbb{R}^n$ onto itself. Then the CP(K,F) is solvable if and only if it is feasible.*

**Proof:** Since K is a polyhedral cone, we have that $K = K^{**}$. Moreover, let $F(x) = q + Mx$, where $M \succeq 0$. Consider the gap program given by

$$\begin{aligned} \text{min} \quad & x^T(Mx + q) \\ \text{subject to} \quad & x \in K \\ & Mx + q \in K^*. \end{aligned}$$

This is a convex QP with an objective that is bounded below by zero on $K$. We then use the Frank-Wolfe theorem - which states that a quadratic function, bounded below on a polyhedral set, attains its minimum on that set. Therefore the dual gap program attains a minimum at $x^*$.

By the necessary conditions of optimality, there exists a $\lambda \in \mathbb{R}^n$ such that

$$K \ni x^* \perp v \equiv q + (M + M^T)x^* - M^T\lambda \in K^*$$
$$K^* \ni q + Mx^* \perp \lambda \in K^{**} = K.$$

Since $(x^*)^T v = 0$ and $\lambda^T v \geq 0$, we have

$$\begin{aligned} 0 &\geq (x^* - \lambda)^T v \\ &= (x^* - \lambda)^T (q + (M + M^T)x^* - M^T\lambda) \\ &= (x^* - \lambda)^T (q + Mx^*) + (x^* - \lambda)^T M^T (x^* - \lambda) \\ &\geq (x^* - \lambda)^T (q + Mx^*) \\ &= (x^*)^T (Mx^* + q). \end{aligned}$$

But $(x^*)^T(Mx^* + q) \geq 0$ implying that $(x^*)^T(Mx^* + q) = 0$ and $x^*$ lies in SOL(K,F).

## General strictly monotone F

**Theorem 6 (Th. 2.4.8)** *Let $K$ be a pointed closed and convex cone in $\mathbb{R}^n$ and $F : K \to \mathbb{R}^n$ be a coninuous map. Consider the following statements*

*(a) F is strictly monotone on K and FEA(K,F) is nonempty;*

*(b) F is strictly monotone on K and CP(K,F) is strictly feasible;*

*(c) the CP(K,F) has a unique solution.*

*It holds that* $(a) \Leftrightarrow (b) \Rightarrow (c)$.

**Proof:**

- Clearly $(b) \implies (a)$
- $(a) \implies (b)$ (omitted) - see Theorem 2.4.8 in FP-I
- Converse is proved in Th. 2.4.8
- $(b) \implies (c)$ follows from Th 2.3.5

## F is affine monotone

Under the assumptions of affineness and monotonicity of F, we obtain the following:

**Lemma 2** *Let $K$ be a convex cone in $\mathbb{R}^n$ and $F(x) = q + Mx$ where $M \succeq 0$. For any $y \in SOL(K, q, M)$ it holds that*

$$SOL(K, q, M) = \{x \in K : q+Mx \in K^*, (M^T+M)(x-y) = 0, q^T(x-y) = 0.\}$$

**Proof:**

- By proposition 2.3.6, we have from monotonicity that $x, y \in SOL(K, q, M)$:

$$(x - y)^T M(x - y) = 0.$$

- For $x \neq y$, we have that $(M + M^T)(x - y) = 0$ which may be further simplified as

$$\begin{aligned} x^T(M + M^T)(x - y) = 0 \\ x^T(M + M^T)x = x^T(M + M^T)y \\ x^T Mx = \tfrac{1}{2}x^T(M + M^T)y \\ \text{similarly } y^T My = \tfrac{1}{2}x^T(M + M^T)y \end{aligned}$$

- Since $x \in SOL(K, F)$, we have that $x^T(q + Mx) = 0 \implies x^T q = -x^T Mx$.

- Similarly, $y^T(q + My) = 0 \implies y^T q = -y^T My$. But

$$y^T My = x^T Mx = \implies x^T q = -x^T Mx = -y^T My = y^T q.$$

Therefore $x$ lies in the set on the right hand side of the specified set equation.

- To prove the reverse inclusion, we need to show that if $x$ lies in

$$\{x \in K : q + Mx \in K^*, (M^T + M)(x - y) = 0, q^T(x - y) = 0\}$$

then $x^T(Mx + q) = 0$. Since $q^T(x - y) = 0, (M + M^T)(x - y) = 0$, we have

$$q^T x = q^T y, (M + M^T)x = (M + M^T)y.$$

- The latter expression implies that $x^T Mx = y^T My$ implying that

$$x^(Mx + q) = y^T(My + y) = 0.$$

The result follows. ∎

**Implication:** The aforementioned lemma implies that $q^T x$ is a constant scalar and $(M + M^T)x$ is a constant vector for all $x \in SOL(K, q, M)$. Therefore if $K$ is polyhedral, then the set given by SOL(K,F) is also polyhedral.

## Polyhedrality of Monotone AVI

- Consider the **affine variational inequality** denoted by AVI(K,q,M) which requires a vector $x$ such that

$$(y - x)^T(Mx + q) \geq 0, \quad \forall y \in K,$$

  where $K$ is a polyhedral set. Note that if $F$ is non necessarily affine, we have a **linearly constrained VI**.

- Before proving a result pertaining to the polyhedrality of the SOL(K,q,M), we provide a result that gives a KKT characterization of the VI.

  **Proposition 1** *Let $K$ be given by*

$$K \equiv \{x \in \mathbb{R}^n : Ax \leq b, Cx = d\}$$

*for some matrices* $A \in \mathbb{R}^{m\times n}$ *and* $B \in \mathbb{R}^{l\times n}$. *A vector* $x$ *solves the VI(K,F) if and only if there exists* $\lambda \in \mathbb{R}^m$ *and* $\mu \in \mathbb{R}^\ell$ *such that*

$$\begin{aligned} F(x) + C^T\mu + A^T\lambda &= 0 \\ Cx - d &= 0 \\ 0 \leq b - Ax \perp \lambda &\geq 0. \end{aligned}$$

**Proof:**

- For a fixed $x$ and K polyhedral, we may formulate the following LP:

$$\begin{aligned} \min \quad & y^T F(x) \\ \text{subject to} \quad & y \in K, \end{aligned}$$

by noting that the VI is equivalent to finding an $x$ such that

$$y^T F(x) \geq x^T F(x) \quad \forall y \in K.$$

- Therefore if $x \in SOL(K, F)$, then $x$ is a solution to this LP. But by LP duality, we have the existence of $(\lambda, \mu)$ such that the following holds:

$$\begin{aligned} F(x) + C^T \mu + A^T \lambda &= 0 \\ Cx - d &= 0 \\ 0 \leq b - Ax \perp \lambda &\geq 0. \end{aligned}$$

- Reverse holds in a similar fashion. ∎

▶ We may now prove the polyhedrality of the solution set of AVI(K,q,M).

Clearly, when K is a cone, this follows from the result for monotone CP(K,q,M).

**Theorem 7** *Let $K$ be polyhedral and $F(x) \equiv Mx + q$. The solution set of AVI(K,q,M) is polyhedral.*

**Proof:** Let K be given as

$$K \equiv \{x \in \mathbb{R}^n : Ax \le b, Cx = d\}$$

for some matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{l \times n}$. However, $x$ is a solution to AVI(K,q,m) if and only if $(x, \mu, \lambda)$ is a solution to

$$\begin{aligned} Mx + q + C^T\mu + A^T\lambda &= 0 \\ Cx - d &= 0 \\ 0 \le b - Ax \perp \lambda &\ge 0. \end{aligned}$$

But this is an mixed LCP (a special case of a monotone affine CP) and has a polyhedral solution set. However, the solution set of AVI(K,q,M) is a projection of this solution set given by the mapping:

$$\begin{pmatrix} x \\ \mu \\ \lambda \end{pmatrix} \in \mathbb{R}^{n+\ell+m} \rightarrow x \in \mathbb{R}^n.$$

Therefore the solution set of AVI(K,q,M) is also polyhedral. ∎