Lecture 7

Monotonicity

September 21, 2008

Outline

- Introduce several monotonicity properties of vector functions
- Are satisfied immediately by gradient maps of convex functions
- In a sense, role of monotonicity in VI/CPs is similar that of convexity in optimization

Focus

- Motivation from convex optimization and game theoretic problems
- Definition of monotone mappings and their variants
- Existence of solutions and convexity of solution sets
- Convex programming - a special case

Monotone functions

Definition 1 *Given a set $K \subseteq \mathbb{R}^n$, a mapping $F : K \to \mathbb{R}^n$ is said to be*

- *(a)* **pseudo-monotone** *on K if for all vectors $x, y \in K$*

$$(x-y)^T F(y) \geq 0 \implies (x-y)^T F(x) \geq 0;$$

- *(b)* **monotone** *on K if for all vectors $x, y \in K$*

$$(x-y)^T (F(x) - F(y)) \geq 0$$

- *(c)* **strictly monotone** *on K if for all vectors $x, y \in K, x \neq y$*

$$(x-y)^T (F(x) - F(y)) > 0$$

- *(d)* ***ξ-monotone*** *on K for some $\xi > 1$ if there exists a $c > 0$ such that for all $x, y \in K$*

$$(x-y)^T(F(x)-F(y)) > c\|x-y\|^{\xi}$$

- *(e)* ***strongly monotone*** *(or 2-monotone) on K if there exists a $c > 0$ such that for all $x, y \in K$*

$$(x-y)^T(F(x)-F(y)) > c\|x-y\|^{2}$$

Relationships

- strongly-monotone $\Longrightarrow$ $\xi-$monotone $\Longrightarrow$ strictly-monotone $\Longrightarrow$ monotone
- monotone $\Longrightarrow$ pseudo-monotone.

In addition, for an affine map, $F(x) \equiv Ax + b$ and $K \equiv \mathbb{R}^n$, we have

- strongly-monotone $\Leftrightarrow$ $\xi-$monotone $\Leftrightarrow$ strictly-monotone $\Leftrightarrow$ $A \succ 0$
- monotonicity $\Leftrightarrow$ $A \succeq 0$

Vector Mappings and their Jacobians

More generally, if $F \in C^1$ on an open convex set $\mathcal{D}$, then the following proposition holds:

Proposition 1 *Let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open convex set, and let $F : \mathcal{D} \to \mathbb{R}^n$ be a continuously differentiable function on $\mathcal{D}$. Then the following hold:*

(a) *F is monotone on $\mathcal{D}$ if and only if $\nabla F(x)$ is positive semidefinite for all $x \in \mathcal{D}$.*

(b) *F is strictly monotone on $\mathcal{D}$ if and only if $\nabla F(x)$ is positive definite for all $x \in \mathcal{D}$.*

(c) *F is strongly monotone on $\mathcal{D}$ if and only if $\nabla F(x)$ is uniformly positive definite over $\mathcal{D}$, i.e., there exists a $c > 0$ such that for all $x \in \mathcal{D}$*

$$y^T \nabla F(x) y \geq c\|y\|^2, \qquad \text{for all } y \in \mathbb{R}^n.$$

Proof: Homework.

Why are monotone mappings important?

- Arise from important classes of optimization/game-theoretic problems

- Can articulate existence/uniqueness statements for such problems

- Convergence properties of algorithms may sometimes (but not always) be restricted to such monotone problems

Convex programming problems

Consider the optimization problem given by

$$\min_{x} \quad \theta(x)$$
$$x \in K,$$

where $K \subseteq \mathbb{R}^n$ is a closed convex set, and $\theta : \mathcal{D} \to \mathbb{R}$ is a twice-continuously differentiable convex function on an open superset set $\mathcal{D}$ of the set K.

- An optimizer of this problem is given by the solution to VI$(K, \nabla\theta)$, i.e., $x \in K$ such that
$$(y - x)^T \nabla\theta(x), \quad \forall y \in K.$$
- When $\theta(\cdot)$ is a convex function, $\nabla^2\theta(x)$ is positive semidefinite everywhere, implying that $\nabla\theta$ is a monotone vector mapping
- Therefore VI$(K, \nabla\theta)$ is a monotone VI
- **NOTE**: This is true even when θ is convex and differentiable

Game-theoretic Problems

Consider the game-theoretic problem given by agent problems:

$$S_i(\mathbf{x}^{-i}) \qquad \min_{x_i} \quad \theta_i(x_i; \mathbf{x}^{-i})$$
$$x_i \in K_i.$$

Proposition 2 *Let K_i be closed and convex sets, and let θ_i be convex and C^1. Then $\mathbf{x}$ is a Nash equilibrium if and only if $\mathbf{x} \in SOL(K, F)$, where*

$$K = \Pi_{i=1}^n K_i \quad \textit{and} \quad \mathbf{F}(x) := (\nabla \theta_i(\mathbf{x}))_{i=1}^n.$$

- K is a Cartesian product of K_i and is also closed and convex

Existence of solutions to monotone VIs

Theorem 3 *Let $K \subseteq \mathbb{R}^n$ be closed convex and $F : K \to \mathbb{R}^n$ be continuous.*

(a) *If F is strictly monotone on K, the VI(K, F) has at most one solution.*

(b) *If F is $\xi-$ monotone on K, the VI(K, F) has a unique solution.*

(c) *If F is Lipschitz continuous and $\xi-$monotone on Ω for some $\xi > 1$, where $K \subseteq \Omega$, then there exists a $c' > 0$ such that*

$$\|x - x^*\| \leq c' \, \|\mathbf{F}_K^{\rm nat}(x)\|^{\frac{1}{\xi-1}} \qquad \text{for every } x \in \Omega,$$

where x^* is the unique solution to the VI(K, F).

Result in (c) provides an upper bound on the distance from the solution

Proof:

(a) Suppose F is strictly monotone on K. Let x, x' be two solutions to VI(K, F). Then for all $y \in K$, we have

$$(y - x)^T F(x) \geq 0 \qquad (y - x')^T F(x') \geq 0.$$

But by substituting x' and x for y in first and second expressions, we obtain

$$(x' - x)^T F(x) \geq 0 \qquad (x - x')^T F(x') \geq 0.$$

But by adding these inequalities, we obtain

$$(x' - x)^T (F(x') - F(x)) \leq 0,$$

contradicting the strict monotonicity of F.

(b) If F is $\xi-$monotone, we merely need to show that VI(K, F) has a solution. Its uniqueness follows from F being strictly monotone. For any $\xi-$monotone mapping with $\xi > 1$, by definition there exists a $c > 0$ such that $\forall x, y \in K$

$$(x-y)^T(F(x)-F(y)) \geq c\|x-y\|^{\xi}$$

For a fixed $y \in K$, it follows that

$$\frac{F(x)^T(x-y)}{\|x\|^{\xi}} \geq \frac{F(y)^T(x-y)}{\|x\|^{\xi}} + c\,\frac{\|x-y\|^{\xi}}{\|x\|^{\xi}}$$

By letting $\|x\| \to \infty$ with $x \in K$, since $\xi > 1$, it follows

$$\liminf_{\substack{x\in K\\ \|x\|\to\infty}} \frac{F(x)^T(x-y)}{\|x\|^{\xi}} \geq c > 0.$$

The existence is now immediate from the following result:

Theorem 4 (Corollary 2.2.6 from FP(I)) *Let* $K \subseteq \mathbb{R}^n$ *be a closed convex set and let* $F : \mathbb{R}^n \to \mathbb{R}^n$ *be continuous. If there exists an* $x^{ref} \in K$ *and a scalar* $\zeta \geq 0$ *such that*

$$\liminf_{x \in K, \|x\| \to \infty} \frac{F(x)^T(x - x^{ref})}{\|x\|^\zeta} > 0, \tag{1}$$

then the VI(K,F) has a nonempty compact solution set.

A seen, F satisfies (1) with $x^{ref} = y$ for an arbitrary $y \in K$ and $\zeta = \xi$.

(c) Let $x^* \in K$ be the solution and $c > 0$ be the constant in ξ-monotonicity For a given $x \in \Omega$, we define $r \equiv \mathbf{F}_K^{\mathrm{nat}}(x)$. Then we have

$$x - r = \Pi_K(x - F(x))$$
$$\Longrightarrow \; (y - x + r)^T(F(x) - r) \geq 0, \quad \forall y \in K$$

with $y = x^*$, we have $\quad (x^* - x + r)^T(F(x) - r) \geq 0$

Since $x^* \in SOL(K, F)$ and $x - r \in K$, it follows

$$(x - r - x^*)^T F(x^*) \geq 0$$

By adding the last two inequalities (after some algebra), we obtain

$$(x - x^*)^T(F(x) - F(x^*)) \leq r^T(F(x) - F(x^*)).$$

The $\xi-$monotonicity and Lipschitz continuity (with constant $L > 0$) of F implies that

$$
\begin{aligned}
c\|x - x^*\|^{\xi} \leq (x - x^*)^T(F(x) - F(x^*)) &\leq L\|r\|\|x - x^*\| \\
\|x - x^*\|^{\xi-1} &\leq \frac{L\|r\|}{c} \\
\|x - x^*\| &\leq c'\|r\|^{\frac{1}{\xi-1}}, \\
\text{where } c' &= \left(\frac{L}{c}\right)^{\frac{1}{\xi-1}}.
\end{aligned}
$$

Generally, strict monotonicity is not sufficient to have a solution

Example : Consider the equation $e^x = 0$ which has no zero on the real line.

Theorem 5 *(Theorem 2.3.4 of FP I) Let $K \subseteq \mathbb{R}^n$ be closed and convex, and $F : K \to \mathbb{R}^n$ be continuous. If F is pseudo-monotone on K, then the three statements* (a), (b), (c) *of the main result of Lecture 6 are equivalent*

Proof: It suffices to show that $(c) \implies (a)$, i.e., that the existence of solution to $VI(K, F)$ implies that, for some $x^{ref} \in K$ the set $L_<$ is bounded.

In particular, we will show that the existence of solution to $VI(K, F)$ implies that the set

$$L_< = \{x \in K \mid F(x)^T(x - x^{ref}) < 0\}$$

is empty for some $x^{ref} \in K$.

Consider a solution x^*, and the set $L_< = \{x \in K \mid F(x)^T(x - x^*) < 0\}$.

Since x^* is solution, we have

$$(x - x^*)^T F(x^*) \geq 0 \qquad \text{for all } x \in K.$$

By pseudo-monotonicity, it follows that

$$(x - x^*)^T F(x) \geq 0 \qquad \text{for all } x \in K,$$

implying that the set $L_<$ is empty. $\square$

Convexity of solution set

Next set of results:

- Solution set of pseudo-monotone VI is always convex

- Sufficient condition for such a VI to have a nonempty bounded solution set

- Need to define recession cones:

  Definition 2 *A recession direction of a set X is a direction d such that for some vector $x \in X$, the ray $\{x + \tau d : \tau \geq 0\}$ is contained in X. The set of all recession directions is denoted by X_∞ and called the recession cone of X.*

Instances of X_∞

- If there is a nonzero w with $w \in X_\infty$, then X_∞ is unbounded

- If X is polyhedral, i.e., $X = \{x : Ax \le b\}$, then

$$X_\infty = \{d \in \mathbb{R}^n : Ad \le 0\}.$$

Theorem 6 *Let $K \subseteq \mathbb{R}^n$ be a closed convex set and $F : K \to \mathbb{R}^n$ be a continuous mapping. Also, let F be pseudo-monotone on K. Then the following hold:*

(a) The solution set $SOL(K, F)$ is convex.

(b) If there exists a vector $x^{ref} \in K$ such that $F(x^{\rm ref})$ belongs to the interior of the dual cone to K_∞, i.e., $F(x^{ref}) \in int(K_\infty)^*$, then $SOL(K, F)$ is nonempty and compact.

Proof:

(a) Let F be pseudo monotone on K. We claim that the solution set SOL(K, F) has the following structure:

$$SOL(K, F) = \cap_{y \in K}\{x \in K : F(y)^T(y - x) \geq 0\}. \qquad (2)$$

We prove this statement by showing that a vector in one of the sets lies in the other.

- Let $x^* \in SOL(K, F)$. Then, we have
$$(y - x^*)^T F(x^*) \geq 0 \quad \forall y \in K.$$
By the pseudo-monotonicity of F on K, we have $F(y)^T(y - x^*) \geq 0, \forall y \in K$, implying that
$$x^* \in \cap_{y \in K}\{x \in K : F(y)^T(y - x) \geq 0.\}$$

- Suppose $x^* \in \cap_{y\in K}\{x \in K : F(y)^T(y-x) \geq 0\}$. Let $z \in K$ be arbitrary. Define $y = \tau x^* + (1-\tau)z$. By convexity of K, y belongs to K for all $\tau \in [0,1]$. Then we have

$$F(y)^T(y-x^*) \geq 0 \quad \equiv \quad F(\tau x^* + (1-\tau)z)^T(z-x^*) \geq 0.$$

Letting $\tau \to 1$, we have

$$F(x^*)^T(z-x^*) \geq 0, \quad \forall z \in K.$$

Hence $x^* \in SOL(K,F)$.

- For a fixed, but arbitrary $y \in K$, the set

$$\{x \in K : F(y)^T(y-x) \geq 0\}$$

is closed and convex. The intersection of any number of convex sets is convex, giving us convexity of SOL(K, F).

(b) We will show that the given property implies that the set

$$L_{\leq} = \{x \in K \mid F(x)^T(x - x^{\mathrm{ref}}) \leq 0\}$$

is bounded. Then, the nonemptyness and compactness of $SOL(K, F)$ follows from Prop. 2.2.3 from FP(I) [main result of Lecture 6].

By the pseudo-monotonicity of F on K, we have that

$$F(x^{\mathrm{ref}})^T(x - x^{\mathrm{ref}}) \leq 0 \quad \forall x \in L_{\leq}$$

Hence, $L_{\leq} \subseteq \{x \in K \mid F(x^{\mathrm{ref}})^T(x - x^{\mathrm{ref}}) \leq 0\} \equiv L$, where the set L is closed and convex. Suppose that L is unbounded, then since $L \subseteq K$, K is also unbounded. Thus, there exists a nonzero recession direction $d \in K_\infty$. For such a direction, we have $F(x^{\mathrm{ref}})^T d \leq 0$.

By the given property, we have $F(x^{\mathrm{ref}}) \in int(K_\infty)^*$. Therefore, there exists a small enough $\delta > 0$ such that

$$F(x^{\mathrm{ref}}) - \delta d \in (K_\infty)^*.$$

Hence

$$0 \leq d^T(F(x^{\mathrm{ref}}) - \delta d) \leq -\delta d^T d < 0.$$

The contradiction implies that K is bounded and since $L_\leq \subseteq K$, we see that $L_\leq$ is bounded.

Additional Properties of Solutions for Pseudo-Monotone VI's

Proposition 7 *[FP-I, Prop 2.3.6] Let* $F : K \to \mathbb{R}^n$ *be pseudo-monotone on a convex set* $K \subseteq \mathbb{R}^n$*. For any two solutions* x^1 *and* x^2 *in SOL(K,F), we have*

$$(x^1 - x^2)^T F(x^1) = (x^1 - x^2)^T F(x_2) = 0.$$

Consequently $(x^1 - x^2)^T (F(x^1) - F(x_2)) = 0$*. In addition** *,*

$$F(SOL(K,F)) \subseteq (SOL(K,F)_\infty)^\perp.$$

Proposition 8 *[FP-I, Cor 2.3.7] Let* $K \subseteq \mathbb{R}^n$ *be closed convex and let* $F : \mathcal{D} \to \mathbb{R}^n$ *be continuously differentiable on the open set* $\mathcal{D}$*, where* $K \subset \mathcal{D}$*. If* F *is monotone on* K *and* $\nabla F(x)$ *is symmetric for all* $x \in K$*, then F(SOL(K,F)) is a singleton.*

*If $x \in SOL(K,F)$ and $d \in (SOL(K,F))_\infty)^\perp$, then $x + d \in SOL(K,F)$.

Application to Convex Optimization

Consider the convex problem given by

$$\begin{aligned} \min \quad & \theta(x) \\ \text{subject to} \quad & x \in K, \end{aligned}$$

where $\theta : \mathcal{D} \to \mathbb{R}$ is a twice-continuously differentiable convex function on a closed convex set $K \subseteq \mathbb{R}^n$.

Proposition 9 *Let $K \subseteq \mathbb{R}^n$ be a closed convex set and $\theta \in C^2$ defined on an open convex set $\mathcal{D}$ containing K. If $S_{opt} \neq \emptyset$, then for any $\bar{x} \in S_{opt}$,*

$$S_{opt} = \{x \in K : \nabla\theta(x) = \nabla\theta(\bar{x}), \nabla\theta(\bar{x})^T(x - \bar{x}) = 0\}.$$

Proof: Let $\bar{S}$ denote $\{x \in K : \nabla\theta(x) = \nabla\theta(\bar{x}), \nabla\theta(\bar{x})^T(x - \bar{x}) = 0\}$. We show that $S_{\rm opt} \subseteq \bar{S}$ by using Propositions 7 and 8.

- Since $\theta(\cdot)$ is convex, $\nabla^2\theta$ is positive-semidefinite over K. Therefore $\nabla\theta$ is monotone over K and the problem may be cast as VI(K,F), with F monotone, and K closed and convex.

- By Proposition 7, we have for any two solutions of the problem $\hat{x}$ and $\bar{x}$,

$$(\hat{x} - \bar{x})^T\nabla\theta(\bar{x}) = (\hat{x} - \bar{x})^T\nabla\theta(x) = 0$$
$$(\hat{x} - \bar{x})^T(\nabla\theta(\hat{x}) - \nabla\theta(\bar{x})) = 0.$$

- To claim $\nabla\theta(\hat{x}) = \nabla\theta(\bar{x})$, we invoke Proposition 8 with $F = \nabla\theta$, which claims that F(SOL(K,F)) is a singleton. Hence $\nabla\theta(\hat{x}) = \nabla\theta(\bar{x})$, implying $\hat{x} \in \bar{S}$.

- We now show that $\bar{S} \subseteq S_{\rm opt}$. Let $x \in \bar{S}$ be arbitrary.

- By the convexity of θ at some $\bar{x} \in S_{\rm opt}$, we have by the gradient inequality

$$\theta(\bar{x}) - \theta(x) \geq (\bar{x} - x)^T \nabla\theta(x).$$

Since $x \in \bar{S}$, we have $\nabla\theta(x) = \nabla\theta(\bar{x})$ and $\nabla\theta(\bar{x})^T(x - \bar{x}) = 0$, implying that

$$\theta(\bar{x}) - \theta(x) \geq (\bar{x} - x)^T \nabla(\bar{x}) = 0.$$

Therefore, $\theta(x) \leq \theta(\bar{x})$ but $\bar{x}$ is optimal solution, i.e.,

$$\theta(\bar{x}) \leq \theta(y) \qquad \text{for all } y \in K.$$

Hence, $\theta(x) = \theta(\bar{x})$, showing that $x \in S_{\rm opt}$.