Lecture 6

Variational Inequalities

Existence of Solutions

September 17, 2008

Outline

- Motivation: Walrasian Equilibrium Problem
- Existence of solutions to a VI (with closed and convex K)
- Existence results: Nash and Walrasian equilibrium problems

Walrasian Equilibrium

The model is used to predict economic activities in a closed economy:

- Determine the production level and the prices of commodities when all interactions between commodities are incorporated
- Production and prices should clear the consumption and the labor market
- Simplified model:
  - n commodities
  - $p \in \mathbb{R}^n$ selling price of commodities; p_j per-unit price of commodity j
  - $b \in \mathbb{R}^n$ initial endowment of commodities; b_j units of commodity j
  - $d(p) : \mathbb{R}^n \to \mathbb{R}^n$ demand; $d_j(p)$ demand function for commodity j
  - m production activities
  - $y \in \mathbb{R}^m$ production level; x_i production level of activity i
  - $c \in \mathbb{R}^m$ production cost; c_i per-unit cost of activity i
  - $A(p) : \mathbb{R}^n \to \mathbb{R}^{m \times n}$ technology input-output model
    - $A(p)p \in \mathbb{R}^m$ per-unit activity return from prices p
    - $A(p)'y \in \mathbb{R}^n$ commodities resulting from activities y

The general (Walrasian) equilibrium problem is equivalent to determining (y, p) such that

$$0 \leq y \perp c - A(p)p \geq 0,$$
$$0 \leq p \perp b + A(p)'y - d(p) \geq 0.$$

Economic interpretation:

- The first complementarity system states that
  - activities are nonnegative
  - and yield nonpositive profits
  - and activities with negative profits are NOT performed
- The second system states that

- Prices are nonnegative
- supplies must satisfy demands
- and excess supplies occur when the good is free ($p_j = 0$)

The demand function is such that $p'd(p) = p'b$ (known as **Walras law**)

Corresponds to a $VI(K, F)$ where K is **not compact**

$$F(y,p) = \begin{bmatrix} c - A(p)p \\ b + A(p)'y - d(p) \end{bmatrix}, \quad K = \{(x,p) \mid x \in \mathbb{R}^m_+,\ p \in \mathbb{R}^n_+\}$$

Note that K is NOT compact but is instead closed and convex. In fact, this property is common to a large number of equilibrium and optimization problems. Q: Does a solution to this VI exist?

Existence of Solutions for Convex Closed K

Establishing the existence of solution reduces to

- Applying degree theory
- Using a classic theorem in topology
  - Every continuous function on a closed set $K \subset \mathbb{R}^n$ has a continuous extension to the entire space
  - General theorem applies to a metric space

Theorem: (**Tietze-Urysohn Extension Theorem**)
Let $\Phi : C \to \mathbb{R}^m$ be a continuous map defined on a closed set $C \subset \mathbb{R}^n$. Then, there exist a continuous extension $\bar{\Phi} : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\bar{\Phi}(x) = \Phi(x) \qquad \text{for all } x \in C$$

Example: In $\mathbb{R}$, interpolate function values linearly

Main Result

Prop. Let $K \subseteq \mathbb{R}^n$ be convex closed set, and let $F : K \to \mathbb{R}^n$ be continuous. Consider the following statements:

(a) There is $\hat{x} \in K$ such that the set

$$L_<(\hat{x}) = \{x \in K \mid F(x)'(x-\hat{x}) < 0\} \qquad \text{is bounded (possibly empty)}$$

(b) There is an open bounded set Ω and a vector $\hat{x} \in K \cap \Omega$ such that

$$F(x)'(x-\hat{x}) \geq 0 \qquad \text{for all } x \in K \cap \mathrm{bd}\Omega$$

(c) The $VI(K,F)$ has a solution

We then have

(1) There holds: (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c)

(2) If the set $L_\leq(\hat{x}) = \{x \in K \mid F(x)'(x-\hat{x}) \leq 0\}$ is nonempty and bounded for some $\hat{x} \in K$, then $SOL(K,F)$ is nonempty and compact

Proof of the Result

(1) *Proof (a)* $\Longrightarrow$ *(b)*. Let the condition (a) hold. Then, there exists an open bounded set Ω containing the set $L_<(\hat{x})$ and the vector $\hat{x}$, so that

$$\hat{x} \in \Omega \qquad \text{and} \qquad L_<(\hat{x}) \cap \mathrm{bd}\Omega = \emptyset.$$

Since $\hat{x} \in K$, it follows $\hat{x} \in K \cap \Omega$. Furthermore, since $L_<(\hat{x}) \cap \mathrm{bd}\Omega = \emptyset$, the relation in (b) follows.

Proof (b) $\Longrightarrow$ *(c)*. Let the condition (b) hold. Let $\bar{F}$ be Tietze-Urysohn continuous extension of F from K to $\mathbb{R}^n$, so that $\bar{F}$ and F agree on K. Therefore,

$$\bar{F}(x)'(x - \hat{x}) \geq 0 \qquad \text{for all } x \in K \cap \mathrm{bd}\Omega.$$

Consider the natural map

$$\bar{F}_K^{\mathrm{nat}}(x) = x - \mathcal{P}_K[x - \bar{F}(x))] \qquad x \in K.$$

Suppose that $\bar{F}_K^{\mathrm{nat}}(x^*) = 0$ for some $x^* \in \mathrm{bd}\Omega$. Then, x^* is a solution to $VI(K, \bar{F})$. Since $\bar{F}$ and F coincide on K, it follows that x^* is a solution to $VI(K, F)$.

Suppose now that $\bar{F}_K^{\rm nat}(x) \neq 0$ for any $x \in {\rm bd}\Omega$. We consider the bounded open set Ω of (b) and the continuous map

$$H(x,t) = x - \mathcal{P}_K[t(x - \bar{F}(x)) + (1-t)\hat{x}] \qquad \text{for } (x,t) \in {\rm cl}\Omega \times [0,1].$$

To apply homotopy invariance principle, we need to show that

$$0 \notin H({\rm bd}\Omega, t) \qquad \text{for any } t \in [0,1].$$

- ($t = 1$:) Note that $H(x,1) = \bar{F}_K^{\rm nat}(x)$, so that
$$0 \notin H({\rm bd}\Omega, 1).$$

- ($t = 0$:) For $t = 0$, we have $H(x,0) = x - \hat{x}$. Note that $H(x,0) = 0$ only when $x = \hat{x}$. But $\hat{x} \in \Omega$, so $\hat{x} \notin {\rm bd}\Omega$. It follows that
$$0 \notin H({\rm bd}\Omega, 0).$$

- ($t \in (0,1)$:) Let $t \in (0,1)$, and let $H(\bar{x},t) = 0$ for some $\bar{x} \in \mathrm{cl}\Omega$. If $\bar{x} = \widehat{x}$, then $\bar{x} \notin \mathrm{bd}\Omega$. So assume that $\bar{x} \neq \widehat{x}$. Then, by definition of H we have

$$\bar{x} = \mathcal{P}_K[t(\bar{x} - \bar{F}(\bar{x})) + (1-t)\widehat{x}].$$

By the projection property, we have

$$(y - \bar{x})' \left(\bar{x} - t(\bar{x} - \bar{F}(\bar{x})) - (1-t)\widehat{x}\right) \geq 0 \quad \text{for all } y \in K.$$

By letting $y = \widehat{x}$ and some algebra, we obtain

$$(\widehat{x} - \bar{x})' \left((1-t)(\bar{x} - \widehat{x}) + t\bar{F}(\bar{x})\right) \geq 0.$$

Hence,

$$(\widehat{x} - \bar{x})'\bar{F}(\bar{x}) \geq \frac{1-t}{t} \|\widehat{x} - \bar{x}\|^2 > 0,$$

implying $(\bar{x} - \widehat{x})'\bar{F}(\bar{x}) < 0$. By condition (b) and the fact F and $\bar{F}$ coincide on K, it follows that $\bar{x} \notin \mathrm{bd}\Omega$. Hence,

$$0 \notin H(\mathrm{bd}\Omega, t) \qquad \text{for any } t \in (0,1).$$

With this and the preceding cases $t = 0$ and $t = 1$, we have established that $0 \notin H(\mathrm{bd}\Omega, t)$ for any $t \in [0,1]$. By the homotopy invariance principle, it follows that $\deg(H(\cdot,t),\Omega,0)$ is independent of t. Note that for $t = 0$, we have $H(x,0) = Ix - \widehat{x}$ so that

$$\deg(H(\cdot,0),\Omega,0) = 1.$$

It follows that

$$\deg(H(\cdot,1),\Omega,0) = 1.$$

Since $H(x,0) = \bar{F}_K^{\mathrm{nat}}(x)$, it follows that the equation $\bar{F}_K^{\mathrm{nat}}(x) = 0$ has a solution. This solution is a solution to $VI(K,\bar{F})$ on K. Since $\bar{F}$ and F coincide on K, this solution is also a solution to $VI(K,F)$.

(2) If the set $L_{\leq}(\widehat{x})$ is nonempty and bounded for some $\widehat{x} \in K$, then since $L_{<}(\widehat{x}) \subset L_{\leq}(\widehat{x})$, the set $L_{<}(\widehat{x})$ is also bounded. Hence $SOL(K,F)$ is nonempty by part (1).
Every solution x^* of $VI(K,F)$ satisfies

$$F(x^*)'(y - x^*) \geq 0 \qquad \text{for all } y \in K.$$

Hence, letting $y = \widehat{x}$, we see that

$$F(x^*)'(x^* - \widehat{x}) \leq 0,$$

showing that $x^* \in L_{\leq}(\widehat{x})$. Hence, $SOL(K,F) \subseteq L_{\leq}(\widehat{x})$, implying by boundedness of $L_{\leq}(\widehat{x})$ that the solution set $SOL(K,F)$ is also bounded. Finally, the set $SOL(K,F)$ is closed by continuity of F, and therefore $SOL(K,F)$ is compact.

Implications: Compact and convex K

Corollary: Let $K \subseteq \mathbb{R}^n$ be compact and convex and let $F : K \rightarrow \mathbb{R}^n$ be continuous. The set SOL(K, F) is nonempty and compact.

Proof: The set $L_{\leq}$ is obviously compact for every $\hat{x} \in K$.

Implications: Closed and convex K

Corollary: Let $K \subseteq \mathbb{R}^n$ be closed convex and let $F : K \to \mathbb{R}^n$ be continuous. Assume that there is a vector $\widehat{x} \in K$ such that

$$F(x)'(x - \widehat{x}) \geq 0 \qquad \text{for all } x \in K.$$

Then, the $VI(K, F)$ has a solution.

Proof: The assumption implies that the set

$$L_<(\widehat{x}) = \{x \in K \mid F(x)'(x - \widehat{x}) < 0\}$$

is empty. Thus, $L_<(\widehat{x})$ is bounded. By the **Existence Theorem** for closed convex K, the $VI(K, F)$ has a solution.

Application to Nash equilibrium problem

Proposition. Let each K_i be a compact convex subset of $\mathbb{R}^n_i$ and each θ_i be continuously differentiable. Suppose that for each fixed tuple $\bar{x}^i$, the function $\theta_i(x^i; \bar{x}^i)$ is convex in x^i. Then the set of Nash equilibrium tuples is nonempty and compact.

Proof. The Nash equilibrium problem is equivalent to VI$(\mathbf{F}, K)$ where

$$K \equiv \Pi_{i=1}^N K_i \quad \text{and} \quad \mathbf{F}(\mathbf{x}) \equiv (\nabla_{x^i} \theta_i(\mathbf{x}))_{i=1}^N.$$

The compactness of K follows from the compactness of each K_i while $\mathbf{F}$ is continuous. By the earlier result, the desired conclusion follows.

Application to Walrasian Equilibrium Problem

Recall that the Walrasian Problem can be casted as $VI(K,F)$ with following identification

$$F(y,p) = \begin{bmatrix} c - A(p)p \\ b + A(p)'y - d(p) \end{bmatrix}, \quad K = \{(x,p) \mid x \in \mathbb{R}^m_+,\ p \in \mathbb{R}^n_+\}$$

Proposition: Let $A(p)$ and $d(p)$ be continuous. Assume that the Walras law holds i.e., $p'd(p) = p'b$. If $c \geq 0$, then Walrasian Equilibrium exists.

Proof: By Walras law, it follows that for any $y \geq 0$ and $p \geq 0$,

$$F(y,p)' \begin{bmatrix} y \\ p \end{bmatrix} = y'c - y'A(p)p + bp - p'A(p)'y - p'd(p) = y'c.$$

Since $c \geq 0$, it follows that

$$F(y,p)' \begin{bmatrix} y \\ p \end{bmatrix} \geq 0 \qquad \text{for all } y \geq 0.$$

Hence, Corollary holds with $\hat{x} = (0,0)$, and the $SOL(K,F)$ is nonempty.

References

- Section 2.2 in Facchinei and Pang's book.