Lecture 5

Variational Inequalities

Existence of Solutions

September 15, 2008
Outline

• Motivating examples:
  • Nash Equilibrium problem
  • Saddle point problem

• Theory for existence results:
  • An introduction to degree theory
  • Fixed-point theorems

• Existence result for $\text{VI}(K, F)$ with $K$ compact

• Examples
Nash Equilibrium Problem: $N$-player game

- Player $i$’s decision is denoted by $x_i$, and $x$ is the vector of these decisions.

- Player $i$’s cost function is $\theta_i(x)$ where $x_i \in K_i$, the $i$th player’s strategy set.

- For each tuple $x_{-i}$, the function $\theta_i(x_i, x_{-i})$ is convex and continuously differentiable in $x_i$. Moreover, the constraint set $K_i$ is convex and compact for each $i$.

- A vector $x^*$ is a Nash equilibrium if and only if $x^*$ solves $VI(K, F)$, where

$$K = K_1 \times \cdots \times K_N, \quad F = \begin{bmatrix} \nabla x_1 \theta_1(x) \\ \vdots \\ \nabla x_N \theta_N(x) \end{bmatrix}$$
- Recall: $x^*$ solves $VI(K, F')$ when $(y - x^*)'F(x^*) \geq 0$ for all $y \in K$

The resulting equilibrium conditions for $G$ are specified by a cartesian VI denoted by $VI(K_i, F_i; \mathcal{N})$ which requires a tuple $\{z_1, \ldots, z_N\}$ such that $z_i$ satisfies

$$VI(K_i, F_i; \mathcal{N}) \quad F_i(z)^T(y_i - z_i) \geq 0, \quad \forall y_i \in K_i, \quad i \in \mathcal{N}.$$  

A simple result sourced from FP-I relates $VI(K_i, F_i; \mathcal{N})$ to a $VI(K, F')$ where $F = (F_i)_{i=1}^N$ and $K = \prod_{i \in \mathcal{N}} K_i$ and is presented next in a modified form.

**Lemma 1 (Prop. 1.4.2 (FP-I))** Let $K_i$ be a closed convex subset of $\mathbb{R}^n$. Then a tuple $z \in SOL(K_i, F_i; \mathcal{N})$ if and only if $z \in SOL(K, F)$ where $K = \prod_{i \in \mathcal{N}} K_i$.

**Proof:**
• (⇒): Suppose $z \in SOL(K_i, F_i; \mathcal{N})$ implying that

$$z_i \in SOL(VI(K_i, F_i(\cdot; z^{-i})))$$

for $i \in \mathcal{N}$. It follows that $\sum_{i=1}^{N}(y_i - z_i)^T F_i(z_i; z^{-i}) \geq 0$ or

$$\begin{pmatrix}
F_1(z_1; z^{-1}) \\
\vdots \\
F_N(z_N; z^{-N})
\end{pmatrix}^T
\begin{pmatrix}
y_1 - z_1 \\
\vdots \\
y_N - z_N
\end{pmatrix} \geq 0, \quad \forall y = [y_i]_{i \in \mathcal{N}} \in K \text{ or } z \in SOL(K, F).
$$

• (⇐): Fix some $i \in \mathcal{N}$. If $z \in SOL(K, F)$ then for $\bar{y} = [\bar{y}_j]_{j \in \mathcal{N}}$ where $\bar{y}_j = z_j$ for $j \neq i$ and $\bar{y}_i \in K_i$ are chosen arbitrarily, we have that

$$(\bar{y}_i - z_i)^T F_i(z) \geq 0.$$
Since this can be repeated for each $i \in \mathcal{N}$, it follows that

$$z \in SOL(K_i, F_i; \mathcal{N}). \square$$
**Saddle Point Problem**

- Given two sets $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, and a function $L(x, y)$, find a pair $(x^*, y^*) \in X \times Y$ such that
  \[ L(x^*, y) \leq L(x^*, y^*) \leq L(x, y^*) \]
  for all $x \in X$, $y \in Y$

- Equivalently: a pair $(x^*, y^*) \in X \times Y$ such that
  \[ L(x^*, y^*) = \min_{x \in X} \max_{y \in Y} L(x, y) = \max_{y \in Y} \min_{x \in X} L(x, y) \]

- We will explore the existence of solutions for the case when
  - $X$ and $Y$ are compact convex sets
  - $L(x, y)$ is continuously differentiable over $X \times Y$
  - $L(\cdot, y)$ convex in $x$ for every $y \in Y$
  - $L(x, \cdot)$ concave in $y$ for every $x \in X$

**Example**: $X = Y = [-1, 1]$, $L(x, y) = x^2 - y^2$, $(0, 0)$ is a saddle point

- Saddle Point Problem as $VI(K, F')$:
  \[ K = X \times Y, \quad F = \begin{bmatrix} \nabla_x L(x, y) \\ -\nabla_y L(x, y) \end{bmatrix} \]
Variational Inequality Problem

We focus on a variational inequality problem $VI(K, F)$:

- Given a set $K$ and a mapping $F : K \to \mathbb{R}^n$, we want to find a point $x^* \in K$ such that
  \[(y - x^*)'F(x^*) \geq 0, \quad \text{for all } y \in K\]

- Main question: When does such a point exist? Is it unique?

- So far you have seen some existence results for the special case of linear complementarity problems

- These can be cast as a variational inequality problem $VI(K, F)$ with

\[K = \{x \mid x \geq 0\}, \quad F(x) = Mx + q\]

- Today's focus is on $VI(K, F)$ where
  - The set $K \subseteq \mathbb{R}^n$ is closed and convex
  - The map $F$ is continuous
Variational Inequality and Projection Problem

Projection Problem:
Given a closed convex set $K \subseteq \mathbb{R}^n$ and a vector $\hat{x} \in \mathbb{R}^n$, find a point in the set $K$ at the smallest distance from $\hat{x}$:

\[ \text{find } x^* \text{ that minimizes } ||x - \hat{x}||^2 \text{ over } x \in K \]

- By the necessary (and sufficient - why?) first-order optimality condition
  \[ (y - x^*)(x^* - \hat{x}) \geq 0 \text{ for all } y \in K \]
- We say $x^*$ is the projection of $\hat{x}$ on the set $K$, which we write
  \[ x^* = P_K[\hat{x}] \]

NOTE: The projection exists because $K$ is closed
The projection is unique because $K$ is convex
• By definition $x^*$ solves $VI(K, F')$ when $x^* \in K$ and
\[(y - x^*)' F(x^*) \geq 0 \quad \text{for all } y \in K\]

• Slight transformation of this relation: $x^* \in K$ solves $VI(K, F')$ when
\[(y - x^*)' (x^* - x^* + F(x^*)) \geq 0 \quad \text{for all } y \in K\]

• We obtain: $x^* \in K$ solves $VI(K, F')$ when $x^*$ is the projection of $x^* - F(x^*)$ on the set $K$, i.e.,
\[x^* \in K \text{ solves } VI(K, F') \quad \text{if and only if} \quad x^* = \mathcal{P}_K[x^* - F(x^*)]\]

• Introducing the mapping $G(x) = \mathcal{P}_K[x - F(x)]$ for $x \in \mathbb{R}^n$, we have
\[x^* \in K \text{ solves } VI(K, F') \quad \text{if and only if} \quad x^* = G(x^*) \quad \text{$x^*$ is a fixed point of $G$}\]

• Letting $\Psi(x) = x - \mathcal{P}_K[x - F(x)]$ for $x \in \mathbb{R}^n$, we have
\[x^* \in \mathbb{R}^n \text{ solves } VI(K, F') \quad \text{if and only if} \quad \Psi(x^*) = 0\]
Degree Theory

• Classical mathematical tool for studying the existence of solutions to an equation of the form

\[ \Psi(x) = p \]

• Under some specific conditions on \( p \) and the mapping \( \Psi \)

• For us of the most interest will be \( p = 0 \) and the mapping \( \Psi \) given by

\[ \Psi(x) = x - P_K[x - F(x)] \]

• This mapping is referred to as a natural map, denoted by \( F_K^{nat}(x) \), i.e.,

\[ F_K^{nat}(x) = x - P_K[x - F(x)] \]

[see section 1.5 in Facchinei and Pang’s book, vol. 1, ]
The Degree Notion in $\mathbb{R}$

The degree notion involves three objects:

- A differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$
- An open bounded interval $(a, b)$
- A value $p$ such that $p \neq f(a)$ and $p \neq f(b)$
- The degree of $f$ over $(a, b)$ at $p$ is given by

$$\deg(f, (a, b), p) = \sum_{\{x \in (a, b) | f(x) = p\}} \text{sgn} \frac{df}{dx}(x)$$
• Note that for $p \notin \text{cl} f((a, b))$, we have $\deg(f, (a, b), p) = 0$.

• Thus, when $\deg(f, (a, b), p) = 0$ we cannot say in general whether the equation $f(x) = p$ has a solution $p \in (a, b)$ or not.

• The degree notion is extended to continuous functions by approximation.
The Degree of a Linear and Affine Maps

- Let $\Phi(x) = Ax$ for some nonsingular matrix $A \in \mathbb{R}^{n \times n}$. Then:
  - For any bounded open set $\Omega$ containing the origin, we have
    \[ \text{deg}(\Phi, \Omega, 0) = \text{sgn} \left( \det A \right) = \pm 1 \]
  - For a given $y$ and any bounded open $\Omega$ set containing $p = A^{-1}y$, we have
    \[ \text{deg}(\Phi, \Omega, p) = \text{sgn} \left( \det A \right) = \pm 1 \]

- Let $\Phi(x) = Ax - b$ for some nonsingular matrix $A \in \mathbb{R}^{n \times n}$ and a vector $b \in \mathbb{R}^n$. Then, for any bounded open set $\Omega$ containing the point $A^{-1}b$, we have
  \[ \text{deg}(\Phi, \Omega, 0) = \text{sgn} \left( \det A \right) = \pm 1 \]

Note: In algebraic topology, the preceding relation is a starting point for defining the degree.

Here, we adopt *axiomatic approach*
Degree Definition

The degree is an integer-valued mapping defined on the collection $\Gamma$ of triples $(\Phi, \Omega, p)$, where

- $\Omega \subseteq \mathbb{R}^n$ is an bounded open set
- $\Phi : \text{cl}(\Omega) \to \mathbb{R}^n$ is continuous
- $p \in \mathbb{R}^n$ is not a critical value: $p \notin \Phi(\text{bd}\Omega)$

Example

$\Phi(x) = \frac{x}{\|x\|}$ for $x \in \mathbb{R}^n, x \neq 0$ and $\Omega = \{x \mid 1 < \|x\| < 2\}$

- Are any of the values $p = e_i$ critical, where $e_i$ is the $i$-th unit vector?
- Is there any non-critical value?
- What are the critical values if $\Omega = \{x \mid 2 < \|x\| < 3\}$?
Def. A mapping \( \deg : \Gamma \to \mathbb{Z} \) [assigns an integer to each \((\Phi, \Omega, p) \in \Gamma\)] is a topological degree if the following three axioms are satisfied:

(A1) For the identity mapping \( I \), we have

\[
\deg(I, \Omega, p) = 1 \quad \text{for any bounded open set } \Omega \text{ and any } p \in \Omega
\]

(A2) For a bounded open set \( \Omega \subseteq \mathbb{R}^n \), and any two disjoint open subsets \( \Omega_1 \subseteq \Omega \) and \( \Omega_2 \subseteq \Omega \), we have additive property

\[
\deg(\Phi, \Omega, p) = \deg(\Phi, \Omega_1, p) + \deg(\Phi, \Omega_2, p) \quad \text{for } p \notin \Phi(\text{cl}\Omega \setminus (\Omega_1 \cup \Omega_2))
\]

(A3) We have the homotopy invariance principle:

\[
\deg(H(\cdot, t), \Omega, v(t)) \quad \text{is independent of } t \in [0, 1]
\]

for any two continuous maps \( H : \text{cl}\Omega \times [0, 1] \to \mathbb{R}^n \) and \( v : [0, 1] \to \mathbb{R}^n \) such that

\[
v(t) \notin H(\text{bd}\Omega, t) \quad \text{for all } t \in [0, 1]
\]
Important Properties

Let $\Omega$ be open bounded subset of $\mathbb{R}^n$. Let $\Phi : \text{cl}\Omega \rightarrow \mathbb{R}^n$ be continuous, and let $p \not\in \Phi(\text{bd}\Omega)$.

Prop. (Space Translation) We have:

- $\deg(\Phi, \Omega, p) = \deg(\Phi - p, \Omega, 0)$
- For any $a \in \mathbb{R}^n$ and $\Phi_a(x) = \Phi(x + a)$,

$$\deg(\Phi_a, \Omega - a, p) = \deg(\Phi, \Omega, p)$$

Theorem (Solution Existence for Equation $\Phi(x) = p$)

- If $\deg(\Phi, \Omega, p) \neq 0$, then there exists an $\hat{x} \in \Omega$ such that $\Phi(\hat{x}) = p$
- If $p \not\in \text{cl}\Phi(\Omega)$, then $\deg(\Phi, \Omega, p) = 0$
Brouwer Fixed-Point Theorem

Theorem: Let $C \subseteq \mathbb{R}^n$ be a compact convex set and let $\Phi : C \rightarrow C$ be a continuous map. Then, the mapping $\Phi$ has a fixed point, i.e.,

$$\Phi(x^*) = x^* \quad \text{for some } x^* \in C.$$  

Proof: We prove the theorem for the case when $C$ is the closed unit ball in $\mathbb{R}^n$, denoted by $B$; the general case follows by homeomorphic mapping. The proof is based on the homotopy invariance principle. Define

$$H(x, t) = x - t\Phi(x) \quad \text{for } (x, t) \in B \times [0, 1].$$

Suppose $H(\hat{x}, \hat{t}) = 0$ for some $(\hat{x}, \hat{t}) \in \text{bd}B \times [0, 1]$. Then, $\hat{x} = \hat{t}\Phi(\hat{x})$, implying $\lVert \hat{x} \rVert = \hat{t}\lVert \Phi(\hat{x}) \rVert$. Since $\hat{x} \in \text{bd}B$, it follows that $\lVert \hat{x} \rVert = 1$, so that $\hat{t} = 1/\lVert \Phi(\hat{x}) \rVert$. We have $\Phi(\hat{x}) \in B$, hence $\hat{t} \geq 1$, implying that $\hat{t} = 1$. Consequently, $H(\hat{x}, \hat{t}) = 0$ with $\hat{t} = 1$ yields $\Phi(\hat{x}) = \hat{x}$, showing
that $\hat{x}$ is a fixed point of $\Phi$ on $B$. Suppose now $0 \neq H(\hat{x}, t)$ for any $(x, t) \in B \times [0, 1]$. Note that $H$ is continuous over its domain. By letting

$$v(t) = 0 \quad \text{for } t \in [0, 1],$$

we see that the homotopy invariance principle applies with $\Omega = \text{int}B$. By this principle, it follows that

$$\deg(H(\cdot, t), \text{int}B, 0) \text{ is independent of } t.$$ 

Therefore, $\deg(H(\cdot, 0), \text{int}B, 0) = \deg(H(\cdot, 1), \text{int}B, 0)$. Since $H(\cdot, 0)$ is the identity, by axiom 1, we have

$$\deg(H(\cdot, 0), \text{int}B, 0) = 1,$$

implying that

$$\deg(H(\cdot, 1), \text{int}B, 0) = 1.$$

Thus by Equation Solution Theorem, we have that there exists an $\hat{x} \in \text{int}B$ such that $H(\hat{x}, 1) = 0$. Therefore, $\hat{x} = \Phi(\hat{x})$, showing that $\Phi$ has a fixed point in $B$.

Homeomorphy between a compact convex set $C$ and the closed unit ball is used to prove the theorem in the general case. $\square$
Definition 1 A mapping $\Phi : S \to T$ is said to be a homeomorphism from $S$ onto $T$ if $\Phi$ is continuous and bijective and $\Phi^{-1} : T \to S$ is also continuous.

- Let $f$ be a function defined on a set $A$ and taking values in a set $B$. Then $f$ is said to be an injection (or injective map, or embedding) if, whenever $f(x) = f(y)$, it must be the case that $x = y$. Equivalently, $x \neq y$ implies $f(x) \neq f(y)$. In other words, $f$ is an injection if it maps distinct objects to distinct objects. An injection is sometimes also called one-to-one.

- Let $f$ be a function defined on a set $A$ and taking values in a set $B$. Then $f$ is said to be a surjection (or surjective map) if, for any $b$ in $B$, there exists an $a \in A$ for which $b = f(a)$. A surjection is sometimes referred to as being "onto."

- A function that is injective and surjective is called bijective.
Existence Result for Compact $C$

**Theorem:**  (Solution Existence for Compact Set)
Let $K \subseteq \mathbb{R}^n$ be a compact convex set, and let $F : K \rightarrow \mathbb{R}^n$ be a continuous map. Then, the solution set of the $VI(K, F)$ is nonempty i.e., $\text{SOL}(K, F) \neq \emptyset$.

**Proof:** Recall that $VI(K, F)$ has a solution if and only if

$$G(x) = P_K[x - F(x)]$$

has a fixed point.

The mapping $G$ is from $K$ to $K$ and is continuous. Thus, by Brouwer Theorem, it has a fixed point in $K$. Hence, $VI(K, F)$ has a solution.$\square$
Implications

- Nash equilibrium exists in $N$-Player Game when for each player $i$: the set $K_i$ is compact convex, $\theta_i(x)$ is continuously differentiable and convex in $x_i$ for each fixed $x_{-i}$

- Nash established the existence of an equilibrium in $N$-player game in 1950 using Kakutani’s fixed-point theorem
- Subsequently, in 1951, he considerably improved the proof by applying directly Brouwer theorem

- Saddle Point Problem has solution when $X$ and $Y$ are compact convex sets, and $L(x, y)$ is continuously differentiable and convex/concave in $x$ and $y$, respectively
Set-Valued Maps

A set-valued map $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ assigns a set $\Phi(x) \subseteq \mathbb{R}^m$ to any $x \in \mathbb{R}^n$. The domain, the range and the graph of the set-valued map $\Phi$ are

$$\text{dom}\Phi = \{x \in \mathbb{R}^n \mid \Phi(x) \neq \emptyset\}$$

$$\text{ran}\Phi = \bigcup_{x \in \text{dom}\Phi} \Phi(x)$$

$$\text{gph}\Phi = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in \Phi(x), \ x \in \text{dom}\Phi\}$$

Def. A set-valued map $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be

- **Closed at point** $\hat{x}$ when for every sequences $\{x_k\} \subset \mathbb{R}^n$ and $\{y_k\} \subset \mathbb{R}^m$ such that $x_k \rightarrow \hat{x}, \ y_k \in \Phi(x_k)$ and $y_k \rightarrow \hat{y}$, we have that $\hat{y} \in \Phi(\hat{x})$

- **Closed on a set** $C$ when $\Phi(x)$ is closed at every point $x$ in $C$
Illustration of a Set-Valued Mapping

Example Consider a closed set $C$ and the set-valued map projection map:

$$\mathcal{P}_C[x] = \left\{ x^* \in C \mid \|x^* - x\| = \min_{z \in C} \|z - x\| \right\}$$
Kakutani’s Fixed-Point Theorem

**Theorem:** Let $C \subseteq \mathbb{R}^n$ be a compact convex set. Let $\Phi : C \rightarrow C$ be a set-valued map such that

- The set $\Phi(x)$ is closed and convex for each $x \in C$;
- The map $\Phi$ is closed on $C$.

Then, $\Phi$ has a fixed point on $C$, i.e.,

$$\hat{x} \in \Phi(\hat{x}) \quad \text{for some } \hat{x} \in C$$
Contraction Mappings

Def. A map $G : X \to \mathbb{R}^n$ for a closed set $X \subseteq \mathbb{R}^n$, is *contraction* on $X$ when there is a scalar $\eta \in (0, 1)$ such that

$$\|G(x) - G(y)\| \leq \eta \|x - y\| \quad \text{for all } x, y \in X.$$

Examples

- $f(x) = \frac{1}{4}x^2$ for $x \in [0, 1]$ is contraction with $\eta = \frac{1}{2}$
- More generally, a differentiable convex function $f : X \to \mathbb{R}^n$ is contraction over a convex closed $X$ with constant $c \in (0, 1)$ when
  $$\|\nabla f(x)\| \leq c \quad \text{for all } x \in X$$
- The projection map $P_X : \mathbb{R}^n \to \mathbb{R}^n$ for a closed and convex set $X$ is **not** a contraction. It is a nonexpansive map provided that the projection is with respect to Euclidean norm:

  $$\|P_X[x] - P_X[y]\| \leq \|x - y\| \quad \text{for all } x, y \in \mathbb{R}^n$$
Fixed-Point Theorem for Contraction

**Theorem:** (Fixed-Point for Contraction)

Let $C \subseteq \mathbb{R}^n$ be a closed set, and let $\Phi : C \rightarrow \mathbb{R}^n$ be a contraction with constant $\eta \in (0, 1)$. Then, the following statements hold:

- the map $\Phi$ has a unique fixed-point $x^*$ in $C$.
- From any starting point $x^0 \in C$, the sequence $\{x^k\}$ converges to $x^*$
- For any such sequence $\{x^k\}$, we have

$$
\|x^k - x^*\| \leq \frac{\eta^k}{1 - \eta} \|x^0 - \Phi(x^0)\|, \quad \forall k \geq 1.
$$

**Proof:** Proof is constructive and it provides an algorithm for actually finding the fixed-point. Also, it provides an estimate of the convergence rate.
• Since \( \Phi : C \rightarrow C \) (maps onto itself), the sequence \( \{x^k\} \) is well-defined.
• If it converges, then limit point is a fixed point of \( \Phi \)
• Since \( \Phi \) is a contraction mapping, uniqueness follows
• Suffices to show that \( \{x^k\} \) converges and the required error inequality holds
• Starting with an arbitrary \( x_0 \in C \), consider a sequence \( \{x_k\} \subseteq C \) generated as follows:

\[
x_{k+1} = \Phi(x_k) \quad \text{for all } k \geq 0
\]

• We show that \( x_k \rightarrow \hat{x} \). Since \( \Phi \) is a contraction, we have for all \( k \)

\[
\|x_{k+1} - x_k\| \leq \eta \|x_k - x_{k-1}\| \leq \cdots \leq \eta^k \|x_1 - x_0\|.
\]

Thus, for any \( k \) and \( m > k \)

\[
\|x_m - x_k\| \leq (\eta^{m-1} + \cdots + \eta^k) \|x_1 - x_0\|
\]
\begin{equation}
\eta^k \left(1 + \cdots + \eta^{m-k-1}\right) \|x_1 - x_0\| \\
\leq \frac{\eta^k}{1 - \eta} \|x_1 - x_0\|
\end{equation}

- This shows that \(\{x^k\}\) is a Cauchy sequence and converges to \(x^*\). If \(m = k + i\) and \(i \to \infty\), then

\[\|x^k - x^*\| \leq \frac{\eta^k}{1 - \eta} \|x^0 - \Phi(x^0)\|, \quad \forall k \geq 1.\]

- It follows that \(x_k \to \hat{x}\) at a geometric rate.
- Uniqueness of this fixed-point follows from \(\Phi\) being contraction.
References

• Facchinei and Pang’s book Chapter 1 and Chapter 2.

  http://www.jstor.org/view/00278424/ap001026/00a00090/0

  http://www.jstor.org/view/0003486x/di961724/96p0127a/0