Lecture 5

Variational Inequalities

Existence of Solutions

September 15, 2008

Outline

- Motivating examples:
  - Nash Equilibrium problem
  - Saddle point problem
- Theory for existence results:
  - An introduction to degree theory
  - Fixed-point theorems
- Existence result for VI(K, F) with K compact
- Examples

Nash Equilibrium Problem: N-player game

- Player i's decision is denoted by x_i, and x is the vector of these decisions
- Player i's cost function is $\theta_i(x)$ where $x_i \in K_i$, the ith player's strategy set
- For each tuple x_{-i}, the function $\theta_i(x_i, x_{-i})$ is convex and continuously differentiable in x_i. Moreover, the constraint set K_i is convex and compact for each i
- A vector x^* is a Nash equilibrium if and only if x^* solves $VI(K, F)$, where

$$K = K_1 \times \cdots \times K_N, \qquad F = \begin{bmatrix} \nabla_{x_1}\theta_1(x) \\ \vdots \\ \nabla_{x_N}\theta_N(x) \end{bmatrix}$$

- Recall: x^* solves $VI(K,F)$ when $(y-x^*)'F(x^*) \geq 0$ for all $y \in K$

The resulting equilibrium conditions for $\mathcal{G}$ are specified by a cartesian VI denoted by VI$(K_i, \mathbf{F}_i; \mathcal{N})$ which requires a tuple $\{z_1, \ldots, z_N\}$ such that z_i satisfies

$$\text{VI}(K_i, \mathbf{F}_i; \mathcal{N}) \qquad \mathbf{F}_i(z)^T(y_i - z_i) \geq 0, \qquad \forall y_i \in K_i, \quad i \in \mathcal{N}.$$

A simple result sourced from FP-I relates VI$(K_i, \mathbf{F_i}; \mathcal{N})$ to a VI$(K, \mathbf{F})$ where $\mathbf{F} = \left(\mathbf{F}_i\right)_{i=1}^{N}$ and $K = \prod K_i$ and is presented next in a modified form.

Lemma 1 (Prop. 1.4.2 (FP-I)) *Let K_i be a closed convex subset of $\mathbb{R}^n$. Then a tuple $z \in SOL(K_i, \mathbf{F}_i; \mathcal{N})$ if and only if $z \in SOL(K, \mathbf{F})$ where $K = \prod_{i \in \mathcal{N}} K_i$.*

Proof:

- ($\Longrightarrow$): Suppose $z \in SOL(K_i, \mathbf{F}_i; \mathcal{N})$ implying that

$$z_i \in SOL(VI(K_i, \mathbf{F}_i(\cdot; z^{-i})))$$

for $i \in \mathcal{N}$. It follows that $\sum_{i=1}^{N} (y_i - z_i)^T \mathbf{F}_i(z_i; z^{-i}) \geq 0$ or

$$\begin{pmatrix} \mathbf{F}_1(z_1; z^{-1}) \\ \vdots \\ \mathbf{F}_N(z_N; z^{-N}) \end{pmatrix}^T \begin{pmatrix} y_1 - z_1 \\ \vdots \\ y_N - z_N \end{pmatrix} \geq 0, \quad \forall y = [y_i]_{i \in \mathcal{N}} \in K \text{ or } z \in SOL(K, \mathbf{F}).$$

- ($\Longleftarrow$): Fix some $i \in \mathcal{N}$. If $z \in SOL(K, \mathbf{F})$ then for $\bar{y} = [\bar{y}_j]_{j \in \mathcal{N}}$ where $\bar{y}_j = z_j$ for $j \neq i$ and $\bar{y}_i \in K_i$ are chosen arbitrarily, we have that

$$(\bar{y}_i - z_i)^T \mathbf{F}_i(z) \geq 0.$$

Since this can be repeated for each $i \in \mathcal{N}$, it follows that

$$z \in SOL(K_i, \mathbf{F}_i; \mathcal{N}).\square$$

Saddle Point Problem

- Given two sets $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, and a function $\mathcal{L}(x, y)$, find a pair $(x^*, y^*) \in X \times Y$ such that

$$\mathcal{L}(x^*, y) \le \mathcal{L}(x^*, y^*) \le \mathcal{L}(x, y^*) \qquad \text{for all } x \in X,\ y \in Y$$

- Equivalently: a pair $(x^*, y^*) \in X \times Y$ such that

$$\mathcal{L}(x^*, y^*) = \min_{x \in X} \max_{y \in Y} \mathcal{L}(x, y) = \max_{y \in Y} \min_{x \in X} \mathcal{L}(x, y)$$

- We will explore the existence of solutions for the case when
  - X and Y are compact convex sets
  - $\mathcal{L}(x, y)$ is continuously differentiable over $X \times Y$
  - $\mathcal{L}(\cdot, y)$ convex in x for every $y \in Y$
  - $\mathcal{L}(x, \cdot)$ concave in y for every $x \in X$

Example: $X = Y = [-1, 1]$, $\mathcal{L}(x, y) = x^2 - y^2$, $(0, 0)$ is a saddle point

- Saddle Point Problem as $VI(K, F)$:

$$K = X \times Y, \qquad F = \begin{bmatrix} \nabla_x \mathcal{L}(x, y) \\ -\nabla_y \mathcal{L}(x, y) \end{bmatrix}$$

Variational Inequality Problem

We focus on a variational inequality problem $VI(K,F)$:

- Given a set K and a mapping $F : K \to \mathbb{R}^n$, we want to find a point $x^* \in K$ such that

$$(y - x^*)'F(x^*) \geq 0, \qquad \text{for all } y \in K$$

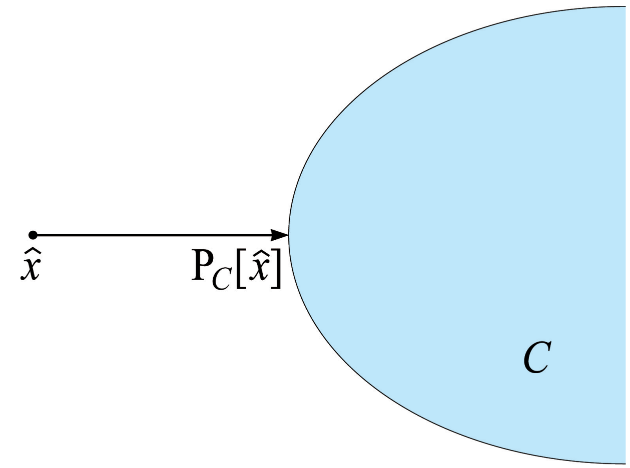
- Main question: When does such a point exist? Is it unique?
- So far you have seen some existence results for the special case of linear complementarity problems
- These can be cast as a variational inequality problem $VI(K,F)$ with

$$K = \{x \mid x \geq 0\}, \qquad F(x) = Mx + q$$

- Todays' focus is on VI(K,F) where
  - The set $K \subseteq \mathbb{R}^n$ is **closed and convex**
  - The map F is **continuous**

Variational Inequality and Projection Problem

Projection Problem:
Given a **closed convex** set $K \subseteq \mathbb{R}^n$ and a vector $\hat{x} \in \mathbb{R}^n$, find a point in the set K at the smallest distance from $\hat{x}$:



$$\text{find } x^* \text{ that minimizes } \quad \|x - \hat{x}\|^2 \quad \text{over } x \in K$$

- By the necessary (and sufficient - why?) first-order optimality condition

$$(y - x^*)'(x^* - \hat{x}) \geq 0 \qquad \text{for all } y \in K$$

- We say x^* is the projection of $\hat{x}$ on the set K, which we write

$$x^* = \mathcal{P}_K[\hat{x}]$$

NOTE: The projection **exists because** K **is closed**
The projection is **unique because** K **is convex**

- By definition x^* solves $VI(K,F)$ when $x^* \in K$ and
$$(y - x^*)' F(x^*) \geq 0 \qquad \text{for all } y \in K$$
- Slight transformation of this relation: $x^* \in K$ solves $VI(K,F)$ when
$$(y - x^*)' \left(x^* - x^* + F(x^*)\right) \geq 0 \qquad \text{for all } y \in K$$
- We obtain: $x^* \in K$ solves $VI(K,F)$ when x^* is the projection of $x^* - F(x^*)$ on the set K, i.e.,

$$x^* \in K \text{ solves } VI(K,F) \quad \textbf{if and only if} \quad x^* = \mathcal{P}_K[x^* - F(x^*)]$$

- Introducing the mapping $G(x) = \mathcal{P}_K[x - F(x)]$ for $x \in \mathbb{R}^n$, we have
$$x^* \in K \text{ solves } VI(K,F) \ \textbf{if and only if} \ x^* = G(x^*)$$
x^* **is a fixed point of** G

- Letting $\Psi(x) = x - \mathcal{P}_K[x - F(x)]$ for $x \in \mathbb{R}^n$, we have
$$x^* \in \mathbb{R}^n \text{ solves } VI(K,F) \quad \textbf{if and only if} \quad \mathbf{\Psi(x^*) = 0}$$

Degree Theory

- Classical mathematical tool for studying the existence of solutions to an equation of the form

$$\Psi(x) = p$$

- Under some specific conditions on p and the mapping Ψ
- For us of the most interest will be $p = 0$ and the mapping Ψ given by

$$\Psi(x) = x - \mathcal{P}_K[x - F(x)]$$

- This mapping is referred to as a **natural map**, denoted by $F_K^{\mathrm{nat}}(x)$, i.e.,

$$F_K^{\mathrm{nat}}(x) = x - \mathcal{P}_K[x - F(x)]$$

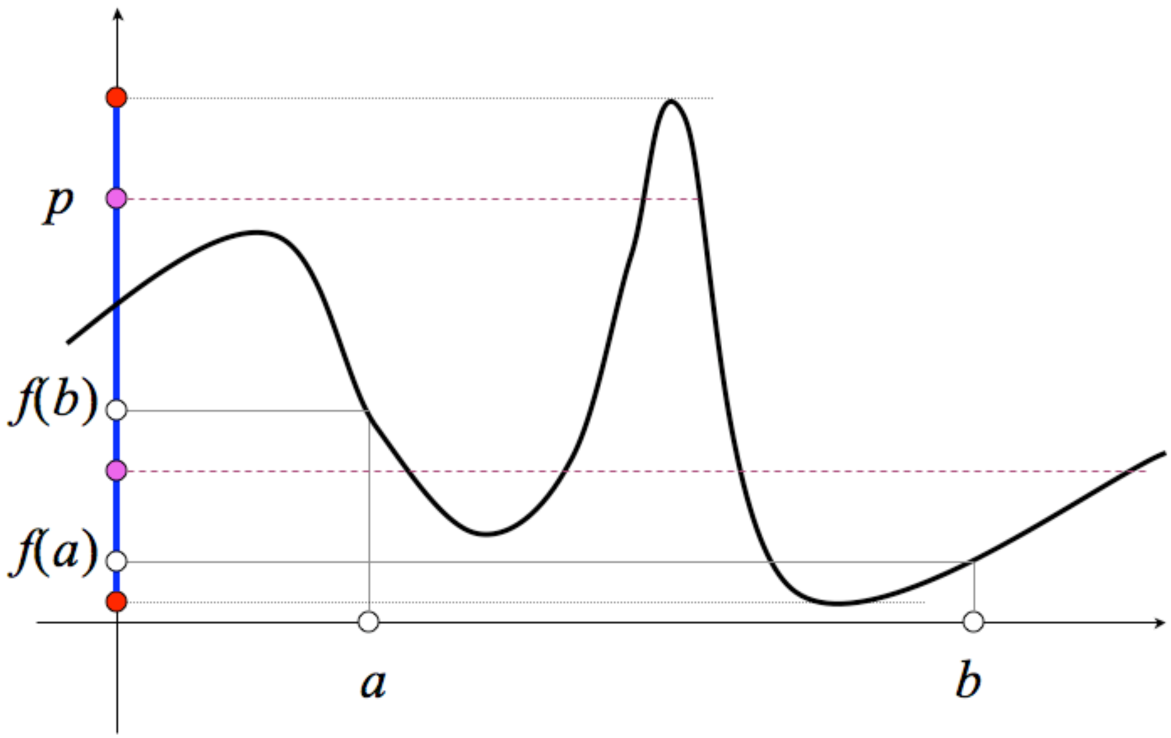
[see section 1.5 in Facchinei and Pang's book, vol. 1,]

The Degree Notion in $\mathbb{R}$

The degree notion involves three objects:

- A differentiable function $f : \mathbb{R} \to \mathbb{R}$
- An open bounded interval (a, b)
- A value p such that $p \neq f(a)$ and $p \neq f(b)$
- The degree of f over (a, b) at p is given by

$$\deg(f, (a, b), p) = \sum_{\{x \in (a,b) \mid f(x) = p\}} \operatorname{sgn} \frac{df}{dx}(x)$$



- Note that for $p \notin \text{cl} f((a,b))$, we have $\deg(f,(a,b),p) = 0$
- Thus, when $\deg(f,(a,b),p) = 0$ we cannot say in general whether the equation $f(x) = p$ has a solution $p \in (a,b)$ or not
- The degree notion is extended to continuous functions by approximation

The Degree of a Linear and Affine Maps

- Let $\Phi(x) = Ax$ for some nonsingular matrix $A \in \mathbb{R}^{n\times n}$. Then:
  - For any bounded open set Ω containing the origin, we have
$$\deg(\Phi, \Omega, 0) = \mathrm{sgn}\,(\mathrm{detA}) = \pm 1$$
  - For a given y and any bounded open Ω set containing $p = A^{-1}y$, we have
$$\deg(\Phi, \Omega, p) = \mathrm{sgn}\,(\mathrm{detA}) = \pm 1$$
- Let $\Phi(x) = Ax - b$ for some nonsingular matrix $A \in \mathbb{R}^{n\times n}$ and a vector $b \in \mathbb{R}^n$. Then, for any bounded open set Ω containing the point $A^{-1}b$, we have
$$\deg(\Phi, \Omega, 0) = \mathrm{sgn}\,(\mathrm{detA}) = \pm 1$$

Note: In algebraic topology, the preceding relation is a starting point for defining the degree

Here, we adopt *axiomatic approach*

Degree Definition

The degree is an integer-valued mapping defined on the collection Γ of triples (Φ, Ω, p), where

- $\Omega \subseteq \mathbb{R}^n$ is an **bounded open** set

- $\Phi : \mathrm{cl}(\Omega) \to \mathbb{R}^n$ is **continuous**

- $p \in \mathbb{R}^n$ is **not a critical value**: $p \notin \Phi(\mathrm{bd}\Omega)$

Example

$\Phi(x) = \frac{x}{\|x\|}$ for $x \in \mathbb{R}^n, x \neq 0$ and $\Omega = \{x \mid 1 < \|x\| < 2\}$

- Are any of the values $p = e_i$ critical, where e_i is the i-th unit vector?
- Is there any non-critical value?
- What are the critical values if $\Omega = \{x \mid 2 < \|x\| < 3\}$?

Def. A mapping $\deg : \Gamma \to Z$ [assigns an integer to each $(\Phi, \Omega, p) \in \Gamma$] is a **topological degree** if the following *three axioms are satisfied*:

(A1) For the **identity mapping** I, we have

$$\deg(I, \Omega, p) = 1 \qquad \text{for any bounded open set } \Omega \text{ and any } p \in \Omega$$

(A2) For a bounded open set $\Omega \subseteq \mathbb{R}^n$, and any two disjoint open subsets $\Omega_1 \subset \Omega$ and $\Omega_2 \subset \Omega$, we have **additive property**

$$\deg(\Phi, \Omega, p) = \deg(\Phi, \Omega_1, p) + \deg(\Phi, \Omega_2, p) \quad \text{for } p \notin \Phi(\mathrm{cl}\Omega \setminus (\Omega_1 \cup \Omega_2))$$

(A3) We have the **homotopy invariance principle**:

$$\deg(H(\cdot, t), \Omega, v(t)) \qquad \text{is independent of } t \in [0, 1]$$

for any two continuous maps $H : \mathrm{cl}\Omega \times [0, 1] \to \mathbb{R}^n$ and $v : [0, 1] \to \mathbb{R}^n$ such that

$$v(t) \notin H(\mathrm{bd}\Omega, t) \qquad \text{for all } t \in [0, 1]$$

Important Properties

Let Ω be open bounded subset of $\mathbb{R}^n$. Let $\Phi : \mathrm{cl}\Omega \to \mathbb{R}^n$ be continuous, and let $p \notin \Phi(\mathrm{bd}\Omega)$.

Prop. (**Space Translation**) We have:

- $\deg(\Phi, \Omega, p) = \deg(\Phi - p, \Omega, 0)$
- For any $a \in \mathbb{R}^n$ and $\Phi_a(x) = \Phi(x + a)$,

$$\deg(\Phi_a, \Omega - a, p) = \deg(\Phi, \Omega, p)$$

Theorem (**Solution Existence for Equation $\Phi(x) = p$**)

- If $\deg(\Phi, \Omega, p) \neq 0$, then there exists an $\hat{x} \in \Omega$ such that $\Phi(\hat{x}) = p$
- If $p \notin \mathrm{cl}\Phi(\Omega)$, then $\deg(\Phi, \Omega, p) = 0$

Brouwer Fixed-Point Theorem

Theorem: Let $C \subseteq \mathbb{R}^n$ be a compact convex set and let $\Phi : C \to C$ be a continuous map. Then, the mapping Φ has a fixed point, i.e.,

$$\Phi(x^*) = x^* \qquad \text{for some } x^* \in C.$$

Proof: We prove the theorem for the case when C is the closed unit ball in $\mathbb{R}^n$, denoted by B; the general case follows by homeomorphic mapping.

The proof is based on the **homotopy invariance principle**. Define

$$H(x,t) = x - t\Phi(x) \qquad \text{for } (x,t) \in B \times [0,1].$$

Suppose $H(\widehat{x},\widehat{t}) = 0$ for some $(\widehat{x},\widehat{t}) \in \mathrm{bd}B \times [0,1]$. Then, $\widehat{x} = \widehat{t}\Phi(\widehat{x})$, implying $\|\widehat{x}\| = \widehat{t}\|\Phi(\widehat{x})\|$. Since $\widehat{x} \in \mathrm{bd}B$, it follows that $\|\widehat{x}\| = 1$, so that $\widehat{t} = 1/\|\Phi(\widehat{x})\|$. We have $\Phi(\widehat{x}) \in B$, hence $\widehat{t} \geq 1$, implying that $\widehat{t} = 1$. Consequently, $H(\widehat{x},\widehat{t}) = 0$ with $\widehat{t} = 1$ yields $\Phi(\widehat{x}) = \widehat{x}$, showing that $\hat{x}$ is a fixed point of Φ on B. Suppose now $0 \neq H(\hat{x}, \hat{t})$ for any $(x, t) \in B \times [0, 1]$. Note that H is continuous over its domain. By letting

$$v(t) = 0 \qquad \text{for } t \in [0, 1],$$

we see that the homotopy invariance principle applies with $\Omega = \mathrm{int}B$. By this principle, it follows that

$$\deg(H(\cdot, t), \mathrm{int}B, 0) \quad \text{is independent of } t.$$

Therefore, $\deg(H(\cdot, 0), \mathrm{int}B, 0) = \deg(H(\cdot, 1), \mathrm{int}B, 0)$. Since $H(\cdot, 0)$ is the identity, by axiom 1, we have

$$\deg(H(\cdot, 0), \mathrm{int}B, 0) = 1,$$

implying that

$$\deg(H(\cdot, 1), \mathrm{int}B, 0) = 1.$$

Thus by Equation Solution Theorem, we have that there exists an $\hat{x} \in \mathrm{int}B$ such that $H(\hat{x}, 1) = 0$. Therefore, $\hat{x} = \Phi(\hat{x})$, showing that Φ has a fixed point in B.

Homeomorphy between a compact convex set C and the closed unit ball is used to prove the theorem in the general case. $\square$

Definition 1 *A mapping* $\Phi : S \to T$ *is said to be a homeomorphism from* S *onto* T *if* Φ *is continuous and bijective and* $\Phi^{-1} : T \to S$ *is also continuous.*

- Let f be a function defined on a set A and taking values in a set B. Then f is said to be an injection (or injective map, or embedding) if, whenever $f(x) = f(y)$, it must be the case that $x = y$. Equivalently, $x \neq y$ implies $f(x) \neq f(y)$. In other words, f is an injection if it maps distinct objects to distinct objects. An injection is sometimes also called one-to-one.

- Let f be a function defined on a set A and taking values in a set B. Then f is said to be a surjection (or surjective map) if, for any b in B, there exists an $a \in A$ for which $b = f(a)$. A surjection is sometimes referred to as being "onto."

- A function that is injective and surjective is called bijective

Existence Result for Compact C

Theorem: (**Solution Existence for Compact Set**)
Let $K \subseteq \mathbb{R}^n$ be a compact convex set, and let $F : K \to \mathbb{R}^n$ be a continuous map. Then, the solution set of the $VI(K, F)$ is nonempty i.e., $\mathrm{SOL(K, F)} \neq \emptyset$.

Proof: Recall that $VI(K, F)$ has a solution if and only if

$$G(x) = \mathcal{P}_K[x - F(x)] \qquad \text{has a fixed point.}$$

The mapping G is from K to K and is continuous. Thus, by Brouwer Theorem, it has a fixed point in K. Hence, $VI(K, F)$ has a solution.□

Implications

- Nash equilibrium exists in N-Player Game when for each player i: the set K_i is compact convex, $\theta_i(x)$ is continuously differentiable and convex in x_i for each fixed x_{-i}
  - Nash established the existence of an equilibrium in N-player game in 1950 using Kakutani's fixed-point theorem
  - Subsequently, in 1951, he considerably improved the proof by applying directly Brouwer theorem

- Saddle Point Problem has solution when X and Y are compact convex sets, and $\mathcal{L}(x, y)$ is continuously differentiable and convex/concave in x and y, respectively

Set-Valued Maps

A set-valued map $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ assigns a set $\Phi(x) \subseteq \mathbb{R}^m$ to any $x \in \mathbb{R}^n$. The **domain**, the **range** and the **graph** of the set-valued map Φ are

$$\mathrm{dom}\Phi = \{x \in \mathbb{R}^n \mid \Phi(x) \neq \emptyset\}$$

$$\mathrm{ran}\Phi = \cup_{x \in \mathrm{dom}\Phi} \Phi(x)$$

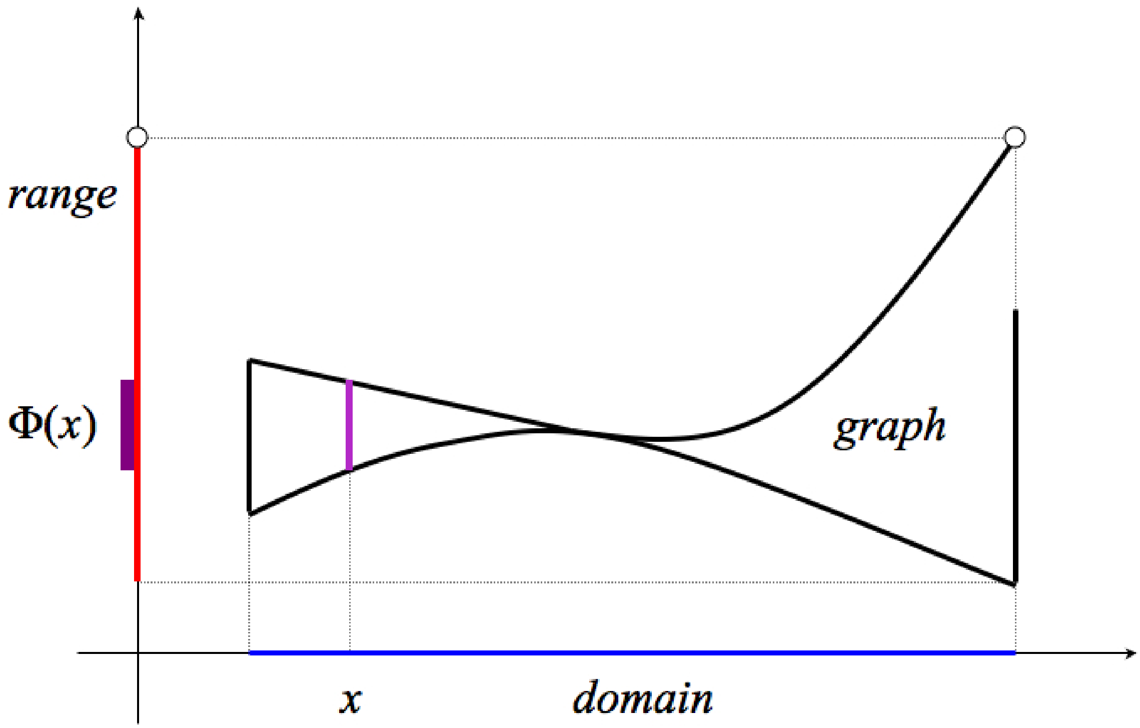
$$\mathrm{gph}\Phi = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in \Phi(x),\ x \in \mathrm{dom}\Phi\}$$

Def. A set-valued map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is said to be

- **Closed at point** $\hat{x}$ when for every sequences $\{x_k\} \subset \mathbb{R}^n$ and $\{y_k\} \subseteq \mathbb{R}^m$ such that $x_k \to \hat{x}$, $y_k \in \Phi(x_k)$ and $y_k \to \hat{y}$, we have that $\hat{y} \in \Phi(\hat{x})$

- **Closed on a set** C when $\Phi(x)$ is closed at every point x in C

Illustration of a Set-Valued Mapping



Example Consider a closed set C and the set-valued map projection map:

$$\mathcal{P}_C[x] = \left\{x^* \in C \mid \|x^* - x\| = \min_{z \in C} \|z - x\|\right\}$$

Kakutani's Fixed-Point Theorem

Theorem: Let $C \subseteq \mathbb{R}^n$ be a compact convex set. Let $\Phi : C \to C$ be a set-valued map such that

- The set $\Phi(x)$ is closed and convex for each $x \in C$;
- The map Φ is closed on C.

Then, Φ has a fixed point on C, i.e.,

$$\hat{x} \in \Phi(\hat{x}) \qquad \text{for some } \hat{x} \in C$$

Contraction Mappings

Def. A map $G : X \to \mathbb{R}^n$ for a closed set $X \subseteq \mathbb{R}^n$, is **contraction** on X when there is a scalar $\eta \in (0, 1)$ such that

$$\|G(x) - G(y)\| \le \eta \, \|x - y\| \qquad \text{for all } x, y \in X.$$

Examples

- $f(x) = \frac{1}{4}x^2$ for $x \in [0, 1]$ is contraction with $\eta = \frac{1}{2}$
- More generally, a differentiable convex function $f : X \to \mathbb{R}^n$ is contraction over a convex closed X with constant $c \in (0, 1)$ when
$$\|\nabla f(x)\| \le c \qquad \text{for all } x \in X$$
- The projection map $\mathcal{P}_X : \mathbb{R}^n \to \mathbb{R}^n$ for a closed and convex set X is **not a contraction**. It is a **nonexpansive map** provided that the projection is with respect to Euclidean norm:

$$\|\mathcal{P}_X[x] - \mathcal{P}_X[y]\| \le \|x - y\| \qquad \text{for all } x, y \in \mathbb{R}^n$$

Fixed-Point Theorem for Contraction

Theorem: (**Fixed-Point for Contraction**)
Let $C \subseteq \mathbb{R}^n$ be a closed set, and let $\Phi : C \to \mathbb{R}^n$ be a contraction with constant $\eta \in (0, 1)$. Then, the following statements hold:

- the map Φ has a unique fixed-point x^* in C.
- From any starting point $x^0 \in C$, the sequence $\{x^k\}$ converges to x^*
- For any such sequence $\{x^k\}$, we have

$$\|x^k - x^*\| \leq \frac{\eta^k}{1-\eta}\|x^0 - \Phi(x^0)\|, \quad \forall k \geq 1.$$

Proof: Proof is constructive and it provides an algorithm for actually finding the fixed-point. Also, it provides an estimate of the convergence rate.

- Since $\Phi : C \to C$ (maps onto itself), the sequence $\{x^k\}$ is well-defined.
- If it converges, then limit point is a fixed point of Φ
- Since Φ is a contraction mapping, uniqueness follows
- Suffices to show that $\{x^k\}$ converges and the required error inequality holds
- Starting with an arbitrary $x_0 \in C$, consider a sequence $\{x_k\} \subseteq C$ generated as follows:

$$x_{k+1} = \Phi(x_k) \qquad \text{for all } k \geq 0$$

- We show that $x_k \to \hat{x}$. Since Φ is a contraction, we have for all k

$$\|x_{k+1} - x_k\| \leq \eta \, \|x_k - x_{k-1}\| \leq \cdots \leq \eta^k \, \|x_1 - x_0\|.$$

  Thus, for any k and $m > k$

$$\|x_m - x_k\| \;\leq\; \left(\eta^{m-1} + \cdots + \eta^k\right) \|x_1 - x_0\|$$

$$= \eta^k \left(1 + \cdots + \eta^{m-k-1}\right) \|x_1 - x_0\|$$
$$\leq \frac{\eta^k}{1-\eta} \|x_1 - x_0\|$$

- This shows that $\{x^k\}$ is a Cauchy sequence and converges to x^*. If $m = k + i$ and $i \to \infty$, then

$$\|x^k - x^*\| \leq \frac{\eta^k}{1-\eta}\|x^0 - \Phi(x^0)\|, \quad \forall k \geq 1.$$

- It follows that $x_k \to \widehat{x}$ at a *geometric rate*.
- Uniqueness of this fixed-point follows from Φ being contraction

References

- Facchinei and Pang's book Chapter 1 and Chapter 2.
- [Nash50] J. Nash, *Equilibrium Points in N-Person Games*, Proc. Nat. Acad. Sci. U.S.A., 36 (1950) 48–49.
http://www.jstor.org/view/00278424/ap001026/00a00090/0
- [Nash51] J. Nash, *Non-Cooperative Games*, The Annals of Mathematics, 2nd Ser., 54 (1951) 286–295.
http://www.jstor.org/view/0003486x/di961724/96p0127a/0