Game theory:
Models, Algorithms and Applications
Lecture 4 Part II
Geometry of the LCP

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Geometry of the Complementarity Problem

• **Definition 1** The set \( \text{pos}(A) \) generated by \( A \in \mathbb{R}^{m \times p} \) represents the convex cone obtained by taking nonnegative linear combinations of the columns of \( A \) or \( \text{pos}(A) := \{ q \in \mathbb{R}^m : q = Av, v \in \mathbb{R}_+^p \} \).

• Therefore if \( q \in \text{pos}(A) \) implies that \( Av = q \) has a nonnegative solution.

• \( \text{pos}(A) \) is also called a finite cone generated by the columns of \( A \).

• Suppose the LCP \( (q, M) \) is written as \( 0 \leq x \perp w \geq 0; w = Mx + q \).

• Then, in solving the LCP problem, we are looking for
  • A representation of \( q \) as an element of the cone \( \text{pos}(I, -M) \)
  • But not using both \( I_{.,i} \) and \( -M_{.,i} \)
Definition 2 [CPS92] Given $M \in \mathbb{R}^{n \times n}$ and $\alpha \subseteq \{1, \ldots, n\}$, we define $C_M(\alpha) \in \mathbb{R}^{n \times n}$ as

$$C_M(\alpha)_{:,i} = \begin{cases} -M_{:,i} & i \in \alpha, \\ I_{:,i} & i \notin \alpha. \end{cases}$$

Specifically

- $C_M(\alpha)$ is a complementary matrix of $M^*$

- $\text{pos}(C_M(\alpha))$ is called the complementary cone (relative to $M$)

- If $C_M(\alpha)$ is nonsingular, then $\text{pos}(C_M(\alpha))$ is said to be full.

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*It may also be called a complementary submatrix of $(I, -M)$. 
For a given $M$,

- There are $2^n$ complementary cones (not necessarily distinct)

- Union of such cones is a cone, denoted by $K(M)$,

$$K(M) = \{q : SOL(q, M) \neq \emptyset\}.$$

- Consider such an object, when $n = 2$

- Let $I_1$ and $I_2$ denote the first and second columns of $I$. Similarly, $M_1$ and $M_2$ represent the columns of $M$.  

Figure 1: Example 1 is to the left, while Example 2 is to the right

Unique solution for all $q$

$K(M) = \mathbb{R}^2$

Multiple solutions for some $q$

$K(M) = \mathbb{R}^2$
Examples 1 and 2

In forthcoming Examples 1–4, the complementary cones are given by
\[ \text{pos}(C_M(\{1, 2\})), \text{pos}(C_M(\{1\})), \text{pos}(C_M(\{2\})) \text{ and pos}(C_M(\emptyset)) \]
In all examples, we have \[ \text{pos}(C_M(\emptyset)) = \mathbb{R}^2_+ \] and
\[ K(M) = \text{pos}(C_M(\{1, 2\})) \cup \text{pos}(C_M(\{1\})) \cup \text{pos}(C_M(\{2\})) \cup \text{pos}(C_M(\emptyset)) \]

- Example 1: \( K(M) = \mathbb{R}^2 \) and every \( q \) lies in exactly one of the complementarity cones - **uniqueness**

- Example 2: \( K(M) = \mathbb{R}^2 \), but \( q \in \mathbb{R}^2_+ \) lies in three complementary cones - **loss of uniqueness**
Figure 2: Example 3 is to the left, while Example 4 is to the right.
Examples 3 and 4

• Example 3:
  • $\text{pos}(C_M(\{1, 2\}))$ is a line (containing both $-M_1$ and $-M_2$
  • Resulting $K(M)$ is a halfspace containing $\mathbb{R}^2_+$
  • If $q \in K(M)$, LCP has a unique solution; no solution otherwise

• Example 4:
  • $-M_1$ is along direction $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ implying that $\text{pos}(C_M(\{1\}))$ is a half-line
  • $K(M)=\text{pos}(C_M(\{2\}))$
  • Every $q$ lies in an even number of complementary cones (possibly zero)
Further geometrical insights

- $\text{pos}(C_M(\emptyset)) = \mathbb{R}^n_+ = \text{pos}(I)$

- $\{\text{pos}(I) \cup \text{pos}(-M)\} \subseteq K(M)$

- $K(M) \subset \text{pos}(I, -M)$, where $\text{pos}(I, -M)$ represents the set of $q$ for which the LCP$(q,M)$ is feasible

- In summary, $\{\text{pos}(I) \cup \text{pos}(-M)\} \subseteq K(M) \subseteq \text{pos}(I, -M)$

- In general $K(M)$ is not convex, but its convex hull viz. $\text{pos}(I, -M)$ always is by definition.
Determining feasibility

- It suffices to check if $q$ belongs to one of the complementary cones

- This in turn requires checking if the following set of systems has a solution

$$C'(\alpha)v = q$$
$$v \geq 0,$$

for some index set $\alpha$.

- Not difficult in principle - however there may be $2^n$ unique index sets - requires doing a phase 1 procedure of an LP

- Definitely need more efficient procedures
The classes $Q$ and $Q_0$

- It was shown that if $M \succ 0$, then LCP$(q,M)$ had a solution for all $q$
- If $M \succeq 0$ and LCP$(q,M)$ was feasible, then LCP$(q,M)$ had a solution
- **Question:** For what classes of matrices do solutions to the LCP always exist? Such a class is denoted by $Q$.
- A partial answer is available - specifically, when is $K(M) \equiv \mathbb{R}^n$? - However, $K(M)$ is often a subset of $\mathbb{R}^n$ and often nonconvex.
- A related question is as follows:
- **Question:** For what classes of matrices do solutions to the LCP exist, when the underlying LCP is feasible? Such a class is denoted by $Q_0$.
- if $M \succeq 0$, then $M \in Q_0$
- We now show an equivalence between $Q_0$ and the convexity of $K(M)$
Equivalence between $Q_0$ and convexity of $K(M)$

**Proposition 1** Let $M \in \mathbb{R}^{n \times n}$. Then the following are equivalent:

1. $M \in Q_0$.
2. $K(M)$ is convex.
3. $K(M) = \text{pos}(I,-M)$

**Proof:**

1. $(1) \implies (2)$: Let $q^1, q^2 \in K(M)$. Therefore $\text{LCP}(q^1, M)$ and $\text{LCP}(q^2, M)$ are solvable. But $\text{LCP}(\lambda q_1 + (1 - \lambda)q_2, M)$ is feasible for all $\lambda \in [0, 1]$.

\[
0 \leq \lambda(Mz_1 + q_1) + (1 - \lambda)(Mz_2 + q_2) = M(\lambda z_1 + (1 - \lambda)z_2) + (\lambda q_1 + (1 - \lambda)q_2) = Mz^\lambda + q^\lambda, \forall \lambda \in [0, 1].
\]
Therefore LCP\((q^\lambda, M)\) is solvable, since \(M \in Q_0\). Hence \(q^\lambda \in K(M)\) and \(K(M)\) is convex.

2. \((2) \implies (3)\): Recall that the convex hull of \(K(M)\) is \(\text{pos}(I, -M)\). If \(K(M)\) is convex, then \(K(M) \equiv \text{pos}(I, -M)\) and the result follows.

3. \((3) \implies (1)\): The cone \(\text{pos}(I, -M)\) contains all vectors \(q\) for which LCP\((q, M)\) is feasible. Therefore if \((3)\) holds, then \(q\) can be generated from one of the complementary cones. In this case, the solution to LCP\((q, M)\) exists; hence, the LCP\((q, M)\) is solvable.
$S$-Matrices

- Consider $S = \{M : \exists z > 0, \ Mz > 0\}$ ($S$ stands for Stiemke)

- It can be seen that $S = \{M : \exists z \geq 0, \ Mz > 0\}$. By continuity of $M$ at $z \geq 0$, we have $M(z + \lambda e) > 0$ for small enough $\lambda > 0$; at the same time, $z + \lambda e > 0$

**Proposition 2** A matrix $M \in \mathbb{R}^n \times \mathbb{R}^n$ is an $S$-matrix if and only if $LCP(q, M)$ is feasible for all $q \in \mathbb{R}^n$

**Proof:** Let $M$ be an $S$-matrix, so that there is a vector $z \geq 0$ such that $Mz > 0$. Then, given any $q$, we can find $\lambda > 0$ large enough so that $\lambda Mz \geq -q$. Thus, $\lambda z$ is feasible for $LCP(q, M)$.

Suppose $LCP(q, M)$ is feasible for any $q$. Choose $\tilde{q} < 0$. Any feasible $z$ for $LCP(\tilde{q}, M)$ satisfies $Mz \geq -\tilde{q} > 0$ and of course $z \geq 0$. Hence, $M$ is an $S$-matrix.
Class $\mathcal{Q}$

In view of Proposition 2, we have

$$\mathcal{Q} = \mathcal{Q}_0 \cap \mathcal{S}$$

- **Checking for $M \in \mathcal{S}$**: Check for feasibility of $\{z : Mz > 0, z > 0\}$ by linear programming (a test with finite termination)

- If we had a finite test for $M \in \mathcal{Q}_0$, then by checking (in a finite time) for $M \in \mathcal{S}$, we would have a finite test for $M \in \mathcal{Q}$

- Unfortunately, no finite test exists for $M \in \mathcal{Q}_0$
Bimatrix games and Copositive Matrices

- The bimatrix game is equivalent to the LCP:

\[
0 \leq \begin{pmatrix} x \\ y \end{pmatrix} \perp \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -e_m \\ -e_n \end{pmatrix} \geq 0.
\]

- Existence and uniqueness of such a solution was left open: Is \( M \in \mathbb{Q} \)?

- Note that \( M \) is not positive semidefinite or positive definite.
Copositive matrices

Definition 3 A matrix $M \in \mathbb{R}^{n \times n}$ is said to be
• copositive if $x^T M x \geq 0$ for all $x \in \mathbb{R}^n_+$.
• strictly copositive if $x^T M x > 0$ for all nonzero $x \in \mathbb{R}^n_+$.
• copositive-plus if $M$ is copositive and the following holds:

\[ z^T M z = 0, z \geq 0 \implies [(M + M^T)z = 0]. \]

• copositive-star if $M$ is copositive and the following holds:

\[ z^T M z = 0, Mz \geq 0, z \geq 0 \implies [M^T z \leq 0]. \]

• Relationship:
Strictly copositive $\subseteq$ copositive-plus $\subseteq$ copositive-star $\subseteq$ copositive
Lemma 1 Let \( M = \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix} \), where \( A, B > 0 \). Then \( M \) is a copositive-plus matrix.

Proof:

- \( M \) is copositive (i.e., \( z^T M z \geq 0 \) for \( z \geq 0 \)): Let \( x, y \geq 0 \). Then

\[
\begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^T Ay + y^T B^T x = x^T (A + B)y.
\]

Since \( x, y \geq 0 \) and \( A, B > 0 \), it follows

\[ x^T (A + B)y \geq 0. \]

Hence, \( M \) is copositive.
• \( M \) satisfies \( [z^T M z = 0, z \geq 0] \implies [(M + M^T)z = 0] \).

Let \( x, y \geq 0 \).

\[
\begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0
\]

\[
\implies x^T (A + B) y = 0
\]

\[
\implies \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} 0 & A + B \\ B^T + A^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.
\]

The last relation and

\[
M + M^T = \begin{pmatrix} 0 & A + B \\ B^T + A^T & 0 \end{pmatrix}, \quad z = \begin{pmatrix} x \\ y \end{pmatrix},
\]

mean that \( z^T (M + M^T) z = 0 \). But \( z \geq 0 \) and \((M + M^T) z \geq 0\) yield \((M + M^T) z = 0\). Hence, \( M \) is copositive-plus.
Proposition 3 Consider an LCP\((q,M)\) with \(M = \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix}\), a copositive plus matrix, and \(q \in \mathbb{R}^n\). Then \(M \in S\) and therefore \(M \in Q\).

Proof: Homework.
Generalizations

- Last two lectures have focused on games which had a specific structure that would allow reformulation as an LCP

- Not always possible since agent problems may have equality constraints (though these can sometimes be transformed - how?)

- **Question**: Can we develop a theory that is less reliant on the precise structure of the agent’s problems

- Our basic framework was:
  - State optimality conditions as an LCP
  - Combine the LCPs obtaining the equilibrium system
  - Use matrix theoretic properties to obtain existence/uniqueness statements
Instead of using complementarity formulations, we may obtain VI formulations of the optimality conditions:

Specifically, player $i$’s optimization problem is given by

\[
\begin{array}{ll}
\text{Player } i \ (x^{-i}) & \text{minimize } \theta_i(x_i, x^{-i}) \\
& \text{subject to } x_i \in X_i,
\end{array}
\]

where $\theta_i(.)$ is in $C^1$ on an open superset of $X_i$, which is a closed convex set of $\mathbb{R}^n$.

$(x_1^*, \ldots, x_N^*)$ is a solution of the Nash game if and only if $x^*$ is a solution
to the set of variational inequalities given by

\[
\begin{align*}
(y_1 - x_1)^T \nabla \theta_1(x_1; x^{-1}) & \geq 0, \quad \forall y_1 \in X_1 \\
(y_2 - x_2)^T \nabla \theta_2(x_2; x^{-2}) & \geq 0, \quad \forall y_2 \in X_2 \\
& \vdots \\
(y_N - x_N)^T \nabla \theta_N(x_N; x^{-N}) & \geq 0, \quad \forall y_N \in X_N,
\end{align*}
\]

or more compactly, \( x^* \) solves the following problem (in \( x \in X \))

\[
(y - x)^T F(x) \geq 0, \quad \forall y \in X = X_1 \times \cdots \times X_N.
\]

- From a geometric standpoint, we have \( x \in SOL(X, F) \) if and only if \( F(x) \) forms a non-obtuse angle with every vector \( y - x \) for \( y \in X \).
This can be related to the normal cone to $X$ at $x$, given by

$$\mathcal{N}_X(x) \equiv \{d \in \mathbb{R}^n : (y-x)^T d \leq 0, \quad \forall y \in X\}.$$  

called the set of normal vectors to $X$ at $x$)

From the statement of the VI, we have to find an $x \in X$ such that

$$(y-x)^T(-F(x)) \leq 0, \quad \forall y \in X$$

or $-F(x)$ is a normal vector to $X$ at $x$; equivalently

$$-F(x) \in \mathcal{N}_X(x) \quad \equiv \quad 0 \in F(x) + \mathcal{N}_X(x).$$
References