Game theory:
Models, Algorithms and Applications
Lecture 4 Part II
Geometry of the LCP

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Geometry of the Complementarity Problem

- **Definition 1** *The set* ***pos***(A) *generated by* $A \in \mathbb{R}^{m\times p}$ *represents the convex cone obtained by taking nonnegative linear combinations of the columns of* A *or* $pos(A) := \{q \in \mathbb{R}^m : q = Av, v \in \mathbb{R}^p_+\}$.
- Therefore if $q \in pos(A)$ implies that $Av = q$ has a nonnegative solution.
- pos(A) is also called a finite cone generated by the columns of A.
- Suppose the LCP(q, M) is written as $0 \le x \perp w \ge 0; w = Mx + q$.
- Then, in solving the LCP problem, we are looking for
  - A representation of q as an element of the cone $pos(I, -M)$
  - But not using both $I_{\cdot,i}$ and $-M_{\cdot,i}$

Definition 2 *[CPS92] Given* $M \in \mathbb{R}^{n\times n}$ *and* $\alpha \subseteq \{1,\ldots,n\}$, *we define* $C_M(\alpha) \in \mathbb{R}^{n\times n}$ *as*

$$C_M(\alpha)_{\cdot,i} = \begin{cases} -M_{\cdot,i} & i \in \alpha, \\ I_{\cdot,i} & i \notin \alpha. \end{cases}$$

Specifically

- $C_M(\alpha)$ is a complementary matrix of M*
- $pos(C_M(\alpha))$ is called the complementary cone (relative to M)
- If $C_M(\alpha)$ is nonsingular, then $pos(C_M(\alpha))$ is said to be *full*.

*It may also be called a complementary submatrix of $(I, -M)$.

For a given M,

- There are 2^n complementary cones (not necessarily distinct)

- Union of such cones is a cone, denoted by K(M),

$$K(M) = \{q : SOL(q, M) \neq \emptyset\}.$$

- Consider such an object, when $n = 2$

- Let I_1 and I_2 denote the first and second columns of I. Similarly, M_1 and M_2 represent the columns of M.

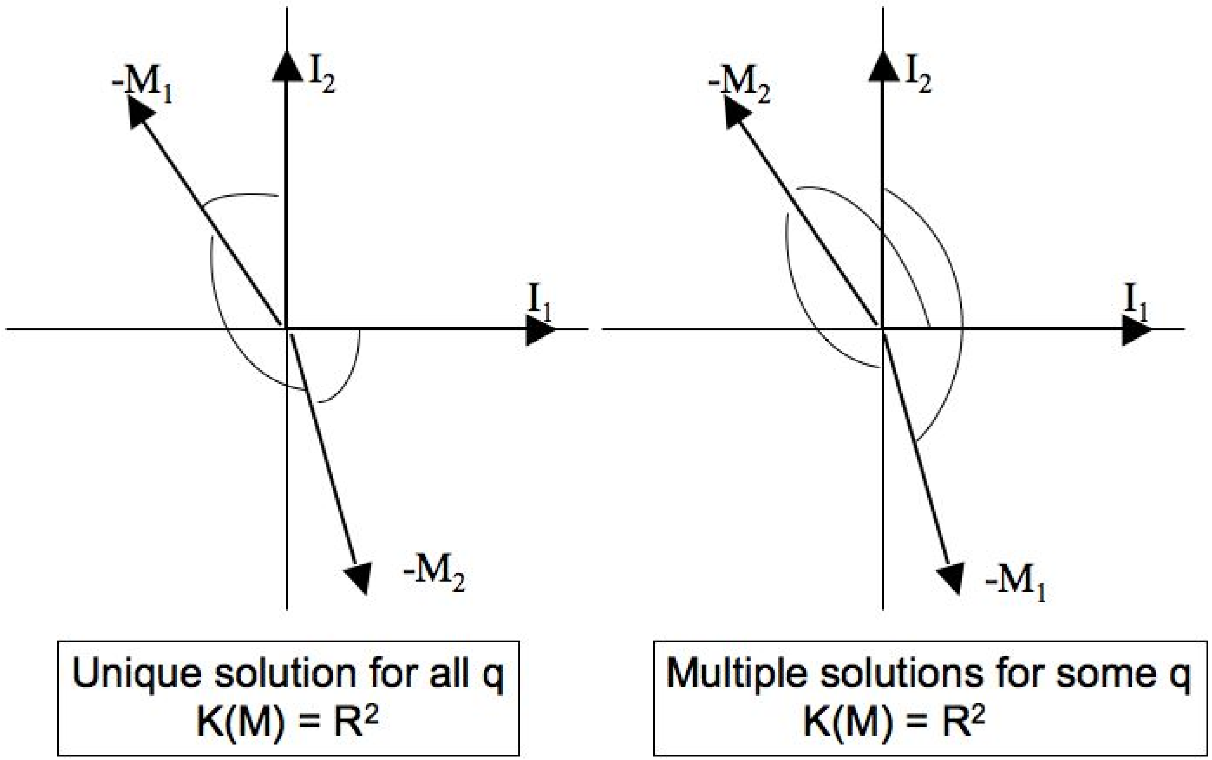

Figure 1: Example 1 is to the left, while Example 2 is to the right

Examples 1 and 2

In forthcoming Examples 1–4, the complementary cones are given by $pos(C_M(\{1,2\}))$, $pos(C_M(\{1\}))$, $pos(C_M(\{2\}))$ and $pos(C_M(\emptyset))$
In all examples, we have $pos(C_M(\emptyset)) = \mathbb{R}^2_+$ and

$$K(M) = pos(C_M(\{1,2\})) \cup pos(C_M(\{1\})) \cup pos(C_M(\{2\})) \cup pos(C_M(\emptyset))$$

- Example 1: K(M)=$\mathbb{R}^2$ and every q lies in exactly one of the complementarity cones - **uniqueness**

- Example 2: K(M)=$\mathbb{R}^2$, but $q \in \mathbb{R}^2_+$ lies in three complementary cones - **loss of uniqueness**

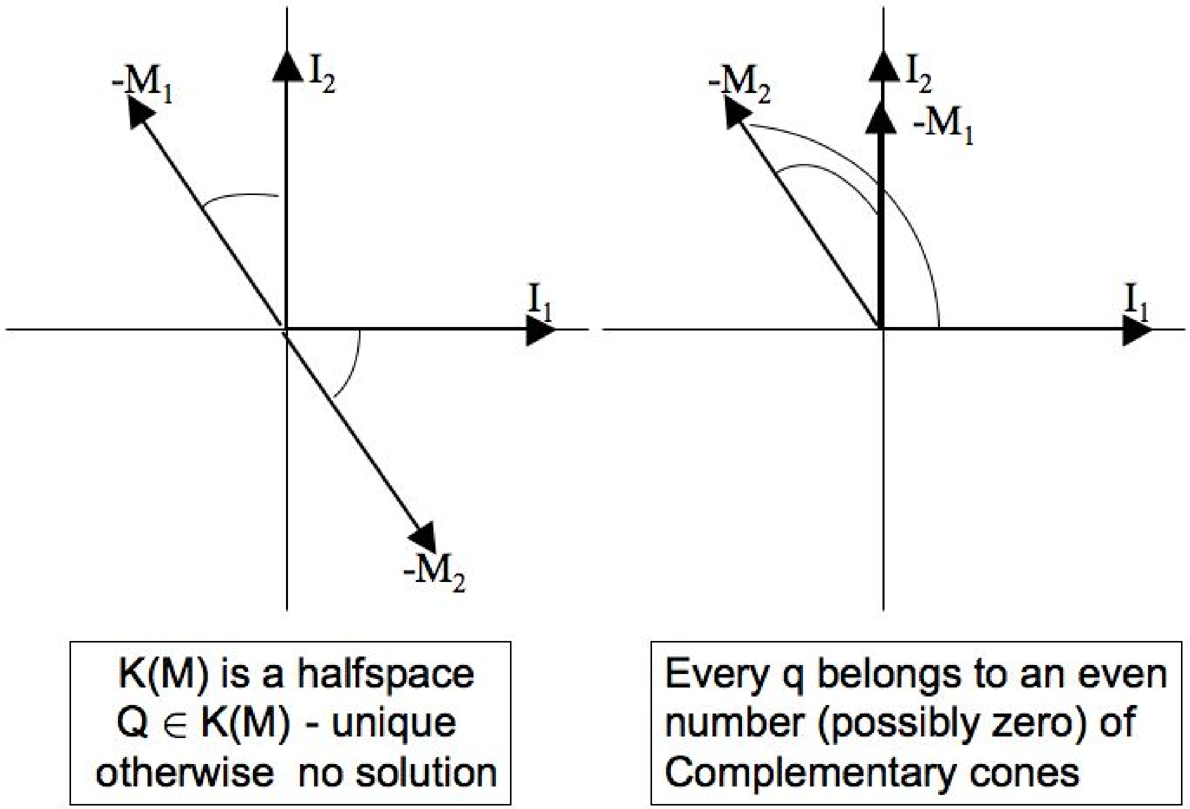

Figure 2: Example 3 is to the left, while Example 4 is to the right

Examples 3 and 4

- Example 3:
  - $pos(C_M(\{1, 2\}))$ is a line (containing both $-M_1$ and $-M_2$
  - Resulting K(M) is a halfspace containing $\mathbb{R}^2_+$
  - If $q \in K(M)$, LCP has a unique solution; no solution otherwise

- Example 4:
  - $-M_1$ is along direction $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ implying that $pos(C_M(\{1\}))$ is a half-line
  - K(M)=$pos(C_M(\{2\}))$
  - Every q lies in an even number of complementary cones (possibly zero)

Further geometrical insights

- pos$(C_M(\emptyset)) = \mathbb{R}^n_+ = pos(I)$
- $\{pos(I) \cup pos(-M)\} \subseteq K(M)$
- $K(M) \subset pos(I, -M)$, where $pos(I, -M)$ represents the set of q for which the LCP(q,M) is feasible
- In summary, $\{pos(I) \cup pos(-M)\} \subseteq K(M) \subseteq pos(I, -M)$
- In general $K(M)$ is not convex, but its convex hull viz. $pos(I, -M)$ always is by definition.

Determining feasibility

- It suffices to check if q belongs to one of the complementary cones
- This in turn requires checking if the following set of systems has a solution

$$\begin{aligned} C(\alpha)v &= q \\ v &\geq 0, \end{aligned}$$

for some index set α.

- Not difficult in principle - however there may be 2^n unique index sets - requires doing a phase 1 procedure of an LP
- Definitely need more efficient procedures

The classes Q and Q_0

- It was shown that if $M \succ 0$, then LCP(q,M) had a solution for all q
- If $M \succeq 0$ and LCP(q,M) was feasible, then LCP(q,M) had a solution
- **Question:** For what classes of matrices do solutions to the LCP always exist? Such a class is denoted by $\mathbf{Q}$.
- A partial answer is availablle - specifically, when is $K(M) \equiv \mathbb{R}^n$? - However, $K(M)$ is often a subset of $\mathbb{R}^n$ and often nonconvex.
- A related question is as follows:
- **Question:** For what classes of matrices do solutions to the LCP exist, when the underlying LCP is feasible? Such a class is denoted by $\mathbf{Q}_0$.
- if $M \succeq 0$, then $M \in \mathbf{Q}_0$
- We now show an equivalence between $\mathbf{Q}_0$ and the convexity of K(M)

Equivalence between $\mathbf{Q}_0$ and convexity of K(M)

Proposition 1 *Let $M \in \mathbb{R}^{n\times n}$. Then the following are equivalent:*

1. *$M \in \mathbf{Q}_0$.*

2. *$K(M)$ is convex.*

3. *$K(M)$ = pos(I,-M)*

Proof:

1. (1) $\implies$ (2): Let $q^1, q^2 \in K(M)$. Therefore LCP(q^1, M) and LCP(q^2, M) are solvable. But LCP$(\lambda q_1 + (1-\lambda)q_2, M)$ is feasible for all $\lambda \in [0, 1]$.

$$\begin{aligned} 0 \leq & \lambda(Mz_1 + q_1) + (1-\lambda)(Mz_2 + q_2) \\ = & M(\lambda z_1 + (1-\lambda)z_2) + (\lambda q_1 + (1-\lambda)q_2) \\ = & Mz^\lambda + q^\lambda, \forall \lambda \in [0, 1]. \end{aligned}$$

Therefore LCP(q^λ, M) is solvable, since $M \in \mathbf{Q}_0$. Hence $q^\lambda \in K(M)$ and K(M) is convex.

2. (2) $\Longrightarrow$ (3): Recall that the convex hull of K(M) is pos(I,-M). If K(M) is convex, then $K(M) \equiv pos(I, -M)$ and the result follows.

3. (3) $\Longrightarrow$ (1): The cone $pos(I, -M)$ contains all vectors q for which LCP(q, M) is feasible. Therefore if (3) holds, then q can be generated from one of the complementary cones. In this case, the solution to $LCP(q, M)$ exists; hence, the $LCP(q, M)$ is solvable.

S-Matrices

- Consider $\mathbf{S} = \{M : \exists\, z > 0,\ Mz > 0\}$ ($\mathbf{S}$ stands for Stiemke)

- It can be seen that $\mathbf{S} = \{M : \exists\, z \geq 0,\ Mz > 0\}$. By continuity of M at $z \geq 0$, we have $M(z + \lambda e) > 0$ for small enough $\lambda > 0$; at the same time, $z + \lambda e > 0$

Proposition 2 *A matrix $M \in \mathbb{R}^n \times \mathbb{R}^n$ is an S-matrix if and only if $LCP(q, M)$ is feasible for all $q \in \mathbb{R}^n$*

Proof: Let M be an S-matrix, so that there is a vector $z \geq 0$ such that $Mz > 0$. Then, given any q, we can find $\lambda > 0$ large enough so that $\lambda Mz \geq -q$. Thus, λz is feasible for $LCP(q, M)$.

Suppose $LCP(q, M)$ is feasible for any q. Choose $\tilde{q} < 0$. Any feasible z for $LCP(\tilde{q}, M)$ satisfies $Mz \geq -\tilde{q} > 0$ and of course $z \geq 0$. Hence, M is an S-matrix.

Class Q

In view of Proposition 2, we have

$$\mathbf{Q} = \mathbf{Q}_0 \cap \mathbf{S}$$

- **Checking for $M \in \mathbf{S}$**: Check for feasibility of $\{z : Mz > 0, z > 0\}$ by linear programming (a test with finite termination)

- If we had a finite test for $M \in \mathbf{Q}_0$, then by checking (in a finite time) for $M \in \mathbf{S}$, we would have a finite test for $M \in \mathbf{Q}$

- Unfortunately, no finite test exists for $M \in \mathbf{Q}_0$

Bimatrix games and Copositive Matrices

- The bimatrix game is equivalent to the LCP:

Bim	$0 \leq \begin{pmatrix} x \\ y \end{pmatrix} \perp \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -e_m \\ -e_n \end{pmatrix} \geq 0.$

- Existence and uniqueness of such a solution was left open: **Is $M \in \mathbf{Q}$?**

- Note that M is not positive semidefinite or positive definite

Copositive matrices

Definition 3 *A matrix $M \in \mathbb{R}^{n\times n}$ is said to be*

- *copositive if $x^T Mx \geq 0$ for all $x \in \mathbb{R}^n_+$.*
- *strictly copositive if $x^T Mx > 0$ for all nonzero $x \in \mathbb{R}^n_+$.*
- *copositive-plus if M is copositive and the following holds:*

$$[z^T Mz = 0, z \geq 0] \implies [(M + M^T)z = 0].$$

- *copositive-star if M is copositive and the following holds:*

$$[z^T Mz = 0, Mz \geq 0, z \geq 0] \implies [M^T z \leq 0].$$

- **Relationship:**
  Strictly copositive $\subseteq$ copositive-plus $\subseteq$ copositive-star $\subseteq$ copositive

Lemma 1 *Let* $M = \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix}$, *where* $A, B > 0$. *Then* M *is a copositive-plus matrix.*

Proof:

- M is copositive (i.e., $z^T M z \geq 0$ for $z \geq 0$): Let $x, y \geq 0$. Then

$$\begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^T A y + y^T B^T x$$
$$= x^T (A + B) y.$$

Since $x, y \geq 0$ and $A, B > 0$, it follows

$$x^T (A + B) y \geq 0.$$

Hence, M is copositive.

- M satisfies $[z^T M z = 0, z \geq 0] \implies [(M + M^T) z = 0]$.
  Let $x, y \geq 0$.

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= 0 \\ \implies \quad x^T (A + B) y &= 0 \\ \implies \quad \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} 0 & A + B \\ B^T + A^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= 0. \end{aligned}$$

The last relation and

$$M + M^T = \begin{pmatrix} 0 & A + B \\ B^T + A^T & 0 \end{pmatrix}, \qquad z = \begin{pmatrix} x \\ y \end{pmatrix},$$

mean that $z^T (M + M^T) z = 0$. But $z \geq 0$ and $(M + M^T) z \geq 0$ yield $(M + M^T) z = 0$. Hence, M is copositive-plus.

Proposition 3 *Consider an LCP(q,M) with* $M = \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix}$*, a copositive plus matrix, and* $q \in \mathbb{R}^n$*. Then* $M \in \mathbf{S}$ *and therefore* $M \in \mathbf{Q}$*.*

Proof: Homework.

Generalizations

- Last two lectures have focused on games which had a specific structure that would allow reformulation as an LCP

- Not always possible since agent problems may have equality constraints (though these can sometimes be transformed - how?)

- **Question**: Can we develop a theory that is less reliant on the precise structure of the agent's problems

- Our basic framework was:
  - State optimality conditions as an LCP
  - Combine the LCPs obtaining the equilibrium system
  - Use matrix theoretic properties to obtain existence/uniqueness statements

- Instead of using complementarity formulations, we may obtain VI formulations of the optimality conditions:

- Specifically, player i's optimization problem is given by

Player i (x^{-i})	$\text{minimize} \quad \theta_i(x_i, \mathbf{x}^{-i})$
	$\text{subject to} \quad x_i \in X_i,$

where $\theta_i(.)$ is in C[1] on an open superset of X_i, which is a closed convex set of $\mathbb{R}^n$.

- $(x_1^*, \ldots, x_N^*)$ is a solution of the Nash game if and only if x^* is a solution to the set of variational inequalities given by

$$\begin{array}{c} (y_1 - x_1)^T \quad \nabla\theta_1(x_1; \mathbf{x}^{-1}) \geq 0, \quad \forall\ y_1 \in X_1 \\ (y_2 - x_2)^T \quad \nabla\theta_2(x_2; \mathbf{x}^{-2}) \geq 0, \quad \forall\ y_2 \in X_2 \\ \vdots \\ (y_N - x_N)^T\ \nabla\theta_N(x_N; \mathbf{x}^{-N}) \geq 0, \quad \forall\ y_N \in X_N, \end{array}$$

or more compactly, x^* solves the following problem (in $x \in X$)

$$(y - x)^T\ F(x) \geq 0, \quad \forall\ y \in X = X_1 \times \cdots \times X_N.$$

- From a geometric standpoint, we have $x \in SOL(X, F)$ if and only if $F(x)$ forms a non-obtuse angle with every vector $y - x$ for $y \in X$.

- This can be related to the normal cone to X at x, given by

$$\mathcal{N}_X(x) \equiv \{d \in \mathbb{R}^n : (y-x)^T d \le 0, \quad \forall y \in X\}.$$

(called the set of normal vectors to X at x)

- From the statement of the VI, we have to find an $x \in X$ such that

$$(y-x)^T(-F(x)) \le 0, \quad \forall y \in X$$

or $-F(x)$ is a normal vector to X at x; equivalently

$$-F(x) \in \mathcal{N}_X(x) \quad \equiv \quad 0 \in F(x) + \mathcal{N}_X(x).$$

References

[CPS92] R. W. Cottle, J-S. Pang, and R. E. Stone. *The Linear Complementarity Problem*. Academic Press, Inc., Boston, MA, 1992.