Game theory:
Models, Algorithms and Applications
Lecture 4 part I
Quadratic Nash games and Linear Complementarity Problems

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Introduction

- Introducing the LCP
- Motivating examples
  - Canonical bimatrix games
  - Nash-Cournot games
- Existence statements of Nash equilibria
  - Positive definiteness
  - Positive semidefiniteness
- Geometry of the LCP (an introduction)

Emphasis of this lecture

- We consider continuous Nash games with quadratic concave utility functions and polyhedral strategy sets
- Specifically, recast equilibrium conditions as pure linear complementarity problems
- Allows for existence/uniqueness statements of original Nash equilibria by analyzing resulting LCPs

An introduction to LCPs

- Consider the constrained convex quadratic program given by

$$\text{QP} \qquad \begin{array}{ll} \underset{x}{\text{minimize}} & \frac{1}{2}x^T M x + q^T x \\ \text{subject to} & x \geq 0 \quad (u) \end{array}$$

where $M \in \mathbb{R}^{n \times n}$, $M \succeq 0$ and symmetric u denotes the dual variables.

- The (sufficient) conditions of optimality are given by

$$\begin{aligned} Mx + q - u &= 0 \\ x, u &\geq 0 \\ [x]_i [u]_i &= 0, \quad i = 1, \ldots, n. \end{aligned}$$

- This may be rewritten as a linear complementarity problem:

Definition 1 *The LCP(q, M) problem* is to determine an x such that*

LCP	$0 \leq x \perp Mx + q \geq 0.$

- We may also write the optimality conditions as a variational inequality problem:

Definition 2 *The VI$(\mathbb{R}^n_+, Mx + q)$ problem is to determine an x such that*

VI	$(Mx + q)^T(y - x) \geq 0, \quad \forall y \in \mathbb{R}^n_+.$

- For every convex problem of the form (QP), the sufficient optimality conditions may be written as a complementarity problem (or VI).

- However, not every LCP can be viewed as the optimality conditions of a QP - specifically, we require that M be symmetric.

$^*w$ is said to be complementary $(\perp)$ to x if $w \perp x \implies w_i x_i = 0 \qquad i = 1, \ldots, n.$

Feasibility and solvability

- The linear complementarity problem has been referred to as the *complementary-pivot problem*, the composite problem amongst others.

- The term LCP first suggested by Cottle

- The LCP problem has associated notions of feasibility and solvability:
  - $\text{FEA}(q, M) = \{x : Mx + q \geq 0, x \geq 0.\}$
  - $\text{SOL}(q, M) = \{x : x \perp w, w = Mx + q, x \in \text{FEA}(q, M)\}$

Relationship to quadratic programming

- If M is a symmetric positive definite matrix, then an equivalence between an unconstrained QP and a related LCP may be drawn

- **Lemma 1** *If M is asymmetric, then $x \in SOL(q, M)$ if and only if x is a global minimizer of (QP)*

$$QP \qquad \begin{array}{ll} \underset{x}{\text{minimize}} & x^T(Mx + q) \\ \text{subject to} & \begin{array}{r} Mx + q \geq 0 \\ x \geq 0, \end{array} \end{array}$$

with a minimizing value of zero.

Special cases

- If $q \geq 0$, then a trivial solution to LCP(q, M) is $\{0\}$.
- If $q \equiv 0$ then LCP$(0, M)$ is called homogenous. In effect, if $z \in$SOL(q, M), then $\lambda z \in SOL(q, M)$, where $\lambda > 0$.
- Question: Clearly $\{0\} \in SOL(0, M)$ but are there any other solutions?

Bimatrix games

- Consider a bimatrix game $\Gamma(A, B)$ in which player 1 (2) has a set of m (n) strategies (called pure in game-theoretic parlance)
- When player 1 chooses strategy i and player 2 chooses strategy j, then the cost incurred by players 1 and 2 are denoted by a_{ij} and b_{ij}, respectively.
- Question: Existence of a Nash equilibrium in pure strategies? In effect, if s_1 and s_2 represent strategies of players 1 and 2, is there is a set (s_1^*, s_2^*) such that

$$s_1^* = \arg\min_i a_{i,s_2^*}, \qquad s_2^* = \arg\min_j b_{s_1^*,j} \ ?$$

- Essentially, each player is faced by a optimization problem with discrete variables

Mixed or randomized strategies

- A **mixed or randomized** strategy is given by a probability distribution over the set of pure strategies

- Specifically, if player 1 and 2 choose distributions x and y, such that

$$X = \left\{x : \sum_{i=1}^{m} x_i = 1,\ x_i \geq 0\right\} \qquad Y = \left\{y : \sum_{i=1}^{n} y_i = 1,\ y_i \geq 0\right\}$$

- Question: Existence of a Nash equilibrium in **mixed** strategies. Specifically, a pair of strategies (x^*, y^*) such that

$$\begin{aligned} (x^*)^T A y^* &\leq x^T A y^*, && \forall x \in X, \\ (x^*)^T B y^* &\leq (x^*)^T B y, && \forall y \in Y. \end{aligned}$$

- (x^*, y^*) is a Nash equilibrium in mixed strategies if neither player can reduce her expected costs by unilaterally changing her strategies.

- Existence of such an equilibrium point proved by Nash [Nas50] (Discuss later)

- Can make A and B positive by adding a large scalar to each element - does not change the equilibrium point

- Consider the LCP

$$\begin{aligned} u = -e_m + Ay \geq 0, \quad x \geq 0, \quad x^T u = 0 \\ v = -e_n + Bx \geq 0, \quad y \geq 0, \quad y^T v = 0, \end{aligned} \tag{1}$$

where $e_m \in \mathbb{R}^m$ and $e_n \in \mathbb{R}^n$ are vectors with all entries equal to 1.

Lemma 2 *If* (x^*, y^*) *is a Nash equilibrium then* (x', y') *is a solution to (1) where*

$$x' = \frac{x^*}{(x^*)^T B y^*}, \qquad y' = \frac{y^*}{(x^*)^T A y^*}.$$

- **Proof:**

$$\begin{aligned}
x^T Ay* &\geq (x^*)^T Ay^* \\
\frac{x^T Ay*}{(x^*)^T Ay*} &\geq 1 \\
\frac{x^T Ay*}{(x^*)^T Ay*} &\geq x^T e_m \\
x^T \left(\frac{Ay*}{(x^*)^T Ay*} - e_m \right) &\geq 0 \\
x^T (Ay' - e_m) &\geq 0 \\
\left(\frac{x}{(x^*)^T By^*} \right)^T (Ay' - e_m) &\geq 0.
\end{aligned}$$

- Moreover $x \geq 0$ and $(x^*)^T By^* > 0$ implying that $x' \geq 0$.
- The matrix A may always be scaled by a positive γ (without affecting

the Nash equilibrium conditions) to ensure that $u = Ay' - e_m \geq 0$.

- $(x')^T u = 0$ can be shown similarly by starting with

$$\begin{aligned}
(x^*)^T Ay* &= (x^*)^T Ay^* \\
\frac{(x^*)^T Ay*}{(x^*)^T Ay*} &= 1 \\
&\vdots \\
(x')^T u &= 0.
\end{aligned}$$

Lemma 3 *Conversely, if* (x', y') *is a solution to (1) then* (x^*, y^*) *is a Nash equilibrium of* $\Gamma(A, B)$ *where*

$$x^* = \frac{x'}{e_m^T x'} \quad y^* = \frac{y'}{e_n^T y'}.$$

The resulting complementarity problem is given by

LCP-Bimatrix	$0 \leq \begin{pmatrix} x \\ y \end{pmatrix} \perp \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -e_m \\ -e_n \end{pmatrix} \geq 0.$

Quadratic Nash Games

- *Nash-Cournot games:* Consider an $N-$ player game in which each player looks at selling quantity x_i at price

$$p(x) := a - m \sum_{i=1}^{N} x_i,$$

where $x = (x_1, \ldots, x_N)$.

- Player i's optimization problem is given by

Cour(x^{-i})	minimize	$c_i x_i - p(x) x_i$
	subject to	$x_i \geq 0,$

- $(x_1^*, \ldots, x_N^*)$ is a solution of the Nash-Cournot game if and only if x^* is a solution to

LCP-Cour	$0 \leq x \perp m(I + ee^T)x + (c - ae) \geq 0.$

where I is the $N \times N$ identity matrix, $c = (c_1, \ldots, c_N)$, and $e \in \mathbb{R}^N$ is the vector of 1's.

- The resulting complementarity problem has M defined as

$$M = m(I + ee^T),$$

which is **symmetric** and $M \succ 0$ for $m > 0$.

Generalizations to Quadratic Nash games

- *Nash-Cournot games:* Consider an $N-$ player game in which player i considers selling quantity x_i at price

$$p(x) := a - m \sum_{i=1}^{N} x_i$$

with a capacity bound b_i and quadratic costs given by $\frac{1}{2} d_i x_i^2 + c_i x_i$.

- Player i's optimization problem is given by

$$\text{Cour}(x^{-i}) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2} d_i x_i^2 + c_i x_i - p(x) x_i \\ \text{subject to} & \begin{array}{l} x_i \leq b_i \quad (\lambda_i) \\ x_i \geq 0. \end{array} \end{array}$$

- The first-order (sufficient) conditions of optimality of player i's problem

$$0 \leq \begin{pmatrix} x_i \\ \lambda_i \end{pmatrix} \perp \begin{pmatrix} 2m + d_i & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_i \\ \lambda_i \end{pmatrix} + \begin{pmatrix} m \sum_{j \neq i} x_i + c_i - a \\ -b_i \end{pmatrix} \geq 0.$$

- $(x_1^*, \ldots, x_N^*)$ is a solution of the Nash-Cournot game if and only if (x^*, λ^*) is a solution to

$$0 \leq \begin{pmatrix} x \\ \lambda \end{pmatrix} \perp \begin{pmatrix} \bar{M} & I \\ I & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} + \begin{pmatrix} c - ae \\ -b \end{pmatrix} \geq 0,$$

where $\bar{M} = \mathrm{diag}(d) + m(I + ee^T)$

- The resulting $\bar{M}$ is positive semidefinite (why?)
- Question: When does an equilibrium to this game exist?

Existence results for an LCP

We begin with a preliminary result [CPS92] :

Lemma 4 *The following results hold:*

- *If the LCP(q,M) is feasible, then the quadratic program (QP) has an optimal solution x^*.*

- *There exists a vector u^* satisfying*

$$0 \leq x^* \perp (M + M^T)x^* + q - M^T u^* \geq 0 \tag{2}$$

$$u^* \geq 0, \quad Mx^* + q \geq 0 \tag{3}$$

$$(u^*)^T(Mx^* + q) = 0. \tag{4}$$

- *The vectors x^* and u^* satisfy*

$$(x^*)^T M^T (x^* - u^*) \leq 0$$
$$(u^*)^T M^T (x^* - u^*) \geq 0$$

Proof:

- Since LCP is feasible, so is QP.

- The QP constraint set $X = \{x : Mx + q \geq 0, x \geq 0\}$ is a polyhedral set (nonempty).

- The quadratic function $x^T(Mx + q) \geq 0$ is bounded below by zero over the polyhedral constraint set X.

- By the Frank-Wolfe theorem, an optimal solution to QP exists, call it x^*.

  Theorem 1 (Frank-Wolfe theorem) *If a quadratic function f is bounded below on a nonempty polyhedron X, then f attains its infimum on X.*

- By the necessary conditions of optimality, there exists a vector of multipliers, denoted by u^*, which together with x^* satisfies the KKT conditions, which are given in Eqs. (2)–(4).

- From $u^* \geq 0$ and $(M + M^T)x^* + q - M^T u^* \geq 0$, it follows

$$(u^*)^T M^T (x^* - u^*) + (u^*)^T (Mx^* + q) \geq 0.$$

  By Eq. (4) we have $(u^*)^T (Mx^* + q) = 0$, implying

$$(u^*)^T M^T (x^* - u^*) \geq 0.$$

- From Eq. (2), we have

$$(x^*)^T M^T (x^* - u^*) = -(x^*)^T (Mx^* + q).$$

In view of $x^* \geq 0$ and $Mx^* + q \geq 0$[see Eqs. (2) and (3)], it follows

$$(x^*)^T M^T (x^* - u^*) \leq 0.$$

Existence Theorem

Theorem 2 *Let M be a positive semidefinite matrix*[†]. *If LCP(q,M) is feasible, then LCP(q,M) is solvable.*

Proof:

- Since LCP is feasible, by Lemma 4 there exist vectors u^* and x^* such that

$$(x^*)^T M^T (x^* - u^*) \leq 0, \quad (5)$$

$$(u^*)^T M^T (x^* - u^*) \geq 0. \quad (6)$$

Therefore, we have

$$(x^* - u^*)^T M^T (x^* - u^*) \leq 0.$$

[†] Not necessarily symmetric

- But $M \succeq 0$ implying that $(x^* - u^*)^T M^T (x^* - u^*) \geq 0$. Hence

$$(x^* - u^*)^T M^T (x^* - u^*) = 0.$$

In view of Eqs. (5) and (6), we have

$$0 \geq (x^*)^T M^T (x^* - u^*) = (u^*)^T M^T (x^* - u^*) \geq 0,$$

implying

$$(x^*)^T M^T (x^* - u^*) = (u^*)^T M^T (x^* - u^*) = 0.$$

- Finally, $x^* \in SOL(q, M)$ can be deduced from Eq. (2), as follows

$$
\begin{aligned}
0 =& (x^*)^T \left((M + M^T)x^* + q - M^T u^* \right) \\
=& (x^*)^T (Mx^* + q) + (x^*)^T M^T (x^* - u^*).
\end{aligned}
$$

Since $(x^*)^T M^T (x^* - u^*) = 0$, we obtain

$$0 = (x^*)^T (Mx^* + q),$$

thus showing that x^* solves the $LCP(q, M)$.

Uniqueness results

Proposition 1 *If $M \in \mathbb{R}^{n\times n}$ is positive-definite, then the LCP(q, M) has a unique solution for all $q \in \mathbb{R}^n$.*

Proof:

- If M is a positive definite matrix then $LCP(q, M)$ is feasible. Specifically, there exists a vector z such that $Mz > 0$ with $z > 0$. This follows from Ville's theorem.

  Lemma 5 (Ville's theorem [CPS92]) *Let $A \in \mathbb{R}^{m\times n}$ be a given matrix. Then $Ax > 0, x > 0$ has a solution if and only if the system*

  $$y^T A \leq 0, \quad y \geq 0, \quad y \neq 0$$

  has no solution.

- We prove that $y^TM \leq 0, y \geq 0, y \neq 0$ has no solution.
  Assume false; then there exists a nonzero y such that $y^TM \leq 0, y \geq 0$. Therefore $y^TMy \leq 0$ contradicting positive definiteness. Hence by Ville's theorem, a vector $z > 0$ with $Mz > 0$ exists.

- **FEA$(q, M) \neq \emptyset$:** Therefore given a z such that $Mz > 0$ and $z > 0$, we may use $\lambda > 0$ large enough so that $\lambda Mz + q \geq 0$. Therefore, λz is a feasible vector for $LCP(q, M)$.

- Existence of a solution now follows by Theorem 2.

- Uniqueness of solution: Given a solution $x^* \in SOL(q, M)$, by Lemma 1 x^* is an optimal solution to

$$\text{QP} \qquad \begin{array}{ll} \text{minimize} & x^T(Mx + q) \\ \text{subject to} & Mx + q \geq 0 \\ & x \geq 0. \end{array}$$

Moreover, $x^T(Mx+q)$ is a strictly convex function, then the QP has a unique global minimizer. Therefore the LCP has a unique solution.

References

[CPS92] R. W. Cottle, J-S. Pang, and R. E. Stone. *The Linear Complementarity Problem*. Academic Press, Inc., Boston, MA, 1992.

[Nas50] J. F. Nash. Equilibrium points in n-person games. *Proceedings of National Academy of Science*, 1950.