Lecture 21
Algorithms for Monotone VIs
Projection Methods

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Outline

- Projection Methods
  - Basic Fixed-Point Iteration
  - Extra-gradient Method
  - Hyperplane Projection Method
Introduction

- Projection methods are conceptually simple methods for solving monotone $VI(K, F')$ for a convex closed set $K$

- Their advantages are
  - Easily implementable and computationally inexpensive
    - When $K$ has structure that makes the projection on $K$ easy
  - Makes them suitable for large scale problems

- Also, they are often used as a sub-procedures in faster and more complex methods (enabling the moves into “promising” regions)

- Their main disadvantage is slow progress since they do not use higher order information
Basic Fixed-Point Iteration

Throughout the rest, we assume that $K \subset \mathbb{R}^n$ is **closed and convex**, and $F : K \rightarrow \mathbb{R}^n$ is **continuous mapping**.

- We consider the “skewed” natural map

$$F_{K,D}^{\text{nat}}(x) = x - \Pi_{K,D}[x - D^{-1}F(x)],$$

where $D$ is a symmetric and positive definite matrix.

- Such a matrix, induces and inner product and a norm in $\mathbb{R}^n$:

$$< x, y >_D = x^T Dy, \quad \| x \| = \sqrt{x^T Dx}.$$

- The skewed projection $\Pi_{K,D}$ on $K$ is just a projection with respect to norm $\| \cdot \|_D$,

$$\hat{x} = \Pi_{K,D}[x] \quad \iff \quad \hat{x} \text{ solves } \min_{y \in K} \| x - y \|_D^2.$$

**Fact**

The projection mapping $x \mapsto \Pi_{K,D}[x]$ is nonexpansive in the norm $\| \cdot \|_D$.
• **Fact** A vector $x^*$ solves $VI(KF)$ if and only if $x^*$ is zero of the skewed natural map, i.e.,

$$x^* = \Pi_{K,D}[x^* - D^{-1}F(x^*)].$$

• Define $\Phi(x) = \Pi_{K,D}[x - D^{-1}F(x)]$ and note that $\Phi : K \rightarrow K$.

• When $D = I$ and $\Phi$ is contraction with respect to Euclidean norm, we have seen that fixed-point method $x_{k+1} = \Phi(x_k)$ produces a sequence with accumulation points being fixed points of $\Phi$ (for any initial $x_0 \in K$).

• The idea of fixed-point here is similar: if we could ensure that $\Phi(x) = \Pi_{K,D}[x - D^{-1}F(x)]$ is a contraction in norm $\| \cdot \|_D$, then we could use “fixed-point” method to find a fixed point of $\Phi(x)$ and hence, a solution to $VI(K,F)$.
Basic Projection Method

Choose a symmetric positive definite matrix $D$. Select an initial vector $x_0 \in K$.

**Step 0:** Set $k = 0$

**Step 1:** Compute $x_{k+1} = \Pi_{K,D}[x_k - D^{-1}F(x_k)]$. If $x_{k+1} = x_k$, stop; we have a solution.

**Step 2:** Otherwise, set $k := k + 1$, and go to Step 1.

In order to ensure the convergence of the sequence $\{x_{k+1}\}$ (or its subsequence) to a fixed point of $\Phi$, we need some conditions of the mapping $F$ and the matrix $D$. 

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Lecture 21

Game theory: Models, Algorithms and Applications
Convergence for Strongly Monotone Mappings

**Theorem 1** Let $F : K \rightarrow \mathbb{R}^n$. Suppose $F$ is strongly monotone and Lipschitz continuous on $K$,

$$(F(x) - F(y))^T(x - y) \geq \mu \|x - y\|^2, \quad \|F(x) - F(y)\| \leq L \|x - y\|^2.$$ 

Also, let

$$\lambda_{\text{max}} < \frac{2\mu}{L^2} \lambda_{\text{min}}^2,$$

where $\lambda_{\text{max}}$ and $\lambda_{\text{min}}$ are the largest and smallest eigenvalues of $D$. Then, the mapping $\Pi_{K,D}[x - D^{-1}F(x)]$ is contraction in $\| \cdot \|_D$ with contraction factor

$$\eta = 1 - \frac{L^2}{\lambda_{\text{max}} \lambda_{\text{min}}^2} \left( \frac{2\mu \lambda_{\text{min}}^2}{L^2} - \lambda_{\text{max}} \right).$$

Therefore, the sequence $\{x_k\}$ converges to the unique solution of $VI(K, F)$. 
Proof: For any two vectors $x, y$ in $K$, we have
\[
\|\Pi_{K,D}[x - D^{-1}F(x)] - \Pi_{K,D}[y - D^{-1}F(y)]\|_D^2 \leq \|x - D^{-1}F(x) - (y - D^{-1}F(y))\|_D^2.
\]
Why? By expanding the last term, we have
\[
\|\Pi_{K,D}[x - D^{-1}F(x)] - \Pi_{K,D}[y - D^{-1}F(y)]\|_D^2 \leq \|x - y\|_D^2 \\
- 2(F(x) - F(y))^T(x - y) + \|F(x) - F(y)\|_{D^{-1}}^2.
\]
Using the strong convexity, we have
\[
(F(x) - F(y))^T(x - y) \geq \mu \|x - y\|^2 \geq \frac{\mu}{\lambda_{\text{max}}} \|x - y\|_D^2. \tag{1}
\]
From
\[
\|F(x) - F(y)\|_{D^{-1}}^2 \leq \frac{1}{\lambda_{\text{min}}} \|F(x) - F(y)\|^2
\]
and Lipschitz continuity of $F$, we obtain
\[
\|F(x) - F(y)\|_{D^{-1}}^2 \leq \frac{L^2}{\lambda_{\text{min}}^2} \|x - y\|_D^2 \tag{2}
\]
By combining the estimates in (1) and (2), we obtain
\[ \| \Pi_{K,D} [x - D^{-1}F(x)] - \Pi_{K,D} [y - D^{-1}F(y)] \|_D^2 \leq \left( 1 - \frac{2\mu}{\lambda_{\text{max}}} + \frac{L^2}{\lambda_{\text{min}}^2} \right) \| x - y \|_D^2. \]

Hence, under the given relation for the eigenvalues of $D$, the mapping $x \mapsto \Pi_{K,D} [x - D^{-1}F(x)]$ is contraction in $\| \cdot \|_D$ and the results follow.

- When $D = \frac{1}{\alpha} I$, the eigenvalue condition reduces to $L^2 < \frac{2\mu}{\alpha}$.
  Thus, if we let $\alpha < \frac{2\mu}{L^2}$, the condition is satisfied.

- The corresponding algorithm becomes
  \[ x_{k+1} = \Pi_K [x_k - \alpha F(x_k)]. \]

- Problem is: we do not always have access to $L$ and $\mu$.
  Hence, we do not know how small $\alpha$ should be to ensure convergence
Co-coercive Mapping

The projection method can be used to solve $VI(K, F')$ with co-coercive mapping. The reason why this work is that the (Euclidean) projection is co-coercive, and when $F$ is co-coercive, then the $\tau$-natural mapping $x \mapsto \Pi_K[x - \tau F(x)]$ is also co-coercive for some range of values of $\tau$. In particular, we have the following results

**Lemma 1** Let $F : K \rightarrow \mathbb{R}^n$ be co-coercive with constant $c$. If $0 < \tau < 4c$, then $F_{K,\tau}^{\text{nat}}(x) = \Pi_K[x - \tau F(x)]$ is co-coercive with constant $1 - \frac{\tau}{4c}$.

**Proof:** See Lemma 12.1.7 of FP-II.
Convergence for Co-coercive Mapping

**Theorem 2** Assume that $VI(K, F)$ has a solution. Let $F : K \rightarrow \mathbb{R}^n$ be co-coercive with constant $c$. Consider the projection method

$$x_{k+1} = \Pi[x_k - \tau F(x_k)],$$

where $\tau < 2c$. Then, the sequence $\{x_k\}$ converges to a solution of $VI(K, F)$.

**Proof** We have for a fixed point $x^*$ of $F_{\text{nat}}^n$ (also a fixed point of $F_{K,\tau}^n$ for any $\tau > 0$),

$$\|x_{k+1} - x^*\|^2 \leq \|\Pi[x_k - \tau F(x_k)] - \Pi[x^* - \tau F(x^*)]\|^2$$

$$\leq \|x_k - x^* - \tau (F(x_k) - F(x^*))\|^2$$

$$= \|x_k - x^*\|^2 - 2\tau (F(x_k) - F(x^*))^T (x - x^*)$$

$$+ \tau^2 \|F(x_k) - F(x^*)\|^2.$$
Since $F$ is co-coercive, we obtain

\[ \|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - \tau(2c - \tau) \|F(x_k) - F(x^*)\|^2 \] (3)

Since $2c - \tau > 0$, it follows that

\[ \text{dist}(x_{k+1}, SOL(K, F)) \leq \text{dist}(x_k, SOL(K, F)) \text{ for all } K. \]

(Recall, that $SOL(K, F)$ is convex and closed here - why?)

Furthermore, by summing the relations in (3) over $k = 0, \ldots, N$ for arbitrary $N$, and letting $N \to \infty$, we obtain

\[ \sum_k \|F(x_k) - F(x^*)\|^2 < \infty, \]
implying $F(x_k) \to F(x^*)$.

(This also implies that $\{F(x^*) \mid x^* \in SOL(K, F)\}$ is a singleton - why would this be expected). In view of the preceding two relations, it follows that

$$\text{dist}(x_k, SOL(K, F)) \to 0.$$ 

Note, also that (3) implies that the scalar sequence $\{\|x_k - x^*\|\}$ is non-increasing for any fixed point $x^*$. Therefore, the scalar sequence $\{\|x_k - x^*\|\}$ is convergent for any fixed point $x^*$. This, and $\text{dist}(x_k, SOL(K, F)) \to 0$ imply that $\{x_k\}$ is convergent and its limit point is in $SOL(K, F)$.

The proof in the FP-II is given in Lemma 12.1.15 for the case when

- $\tau$ is varying and $\tau_k F(x_k)$ is replaced with $F^k(x_k)$, where all $VI(K, F_k)$ have the same solution set.
- $K = \mathbb{R}^n$
- Assuming that each mapping $F^k : \mathbb{R}^n \to \mathbb{R}^n$ is co-coercive with $c_k$, and the following condition is satisfied: $\inf_k c_k > 1/2$. 
**Extra-Gradient Method**

The name of method comes from its equivalent version in optimization ($F = \nabla f(x)$).

It has two projection steps

$$z_{k+1} = \Pi_K [x - \tau F(x_k)], \quad x_{k+1} = \Pi_K [x_k - \tau F(z_{k+1})]$$

The main iterate is $x_{k+1}$. The extra-iterate $z_{k+1}$ is used to construct the direction for moving away from $x_k$.

- The advantage of taking an extra step is that the algorithm performs better than the projection method
- The convergence analysis still requires $F$ to be Lipschitz
- It can be used to solve $VI(K, F)$ with a pseudo-monotone map $F$
- We will study it as applied to a monotone $VI(K, F)$
Basic Iterate Relation

**Theorem 3** Let $F$ be monotone and Lipschitz continuous on $K$ with constant $L$. Let $x^*$ be a solution of $VI(K,F)$. Then, for all $k$ we have

$$
\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - (1 - \tau^2 L^2) \|z_{k+1} - x_{k+1}\|^2.
$$

**Proof** For a projection on convex closed set $K$, we have for any $x \in \mathbb{R}^n$,

$$
\|\Pi_K[x] - z\|^2 \leq \|x - z\|^2 - \|\Pi_K[x] - x\|^2 \quad \text{for all } z \in K.
$$
(can be seen by a more careful analysis in Projection Theorem).

Using this relation with $x = x_{k+1}$ and $z = x^*$, we obtain

$$
\|x_{k+1} - x^*\|^2 \leq \|x_k - \tau F(z_{k+1}) - x^*\|^2 - \|x_{k+1} - (x_k - \tau F(z_{k+1}))\|^2.
$$

By expanding the terms on the right hand side, we have

$$
\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - \|x_{k+1} - x_k\|^2 + 2\tau F(z_{k+1})^T(x^* - x_{k+1}). \quad (4)
$$
By the monotonicity of $F$ and $x^* \in SOL(K, F)$, it follows

$$(F(z_{k+1}) - F(x^*))^T(z_{k+1} - x^*) \geq 0 \quad \implies \quad F(z_{k+1})^T(z_{k+1} - x^*) \geq 0.$$ 

Hence, $F(z_{k+1})^T(z_{k+1} - x_{k+1}) + F(z_{k+1})^T(x_{k+1} - x^*) \geq 0$ implying that

$$F(z_{k+1})^T(z_{k+1} - x_{k+1}) \geq F(z_{k+1})^T(x^* - x_{k+1})$$

Using this relation in (4), we see

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - \|x_{k+1} - x_k\|^2 + 2\tau F(z_{k+1})^T(z_{k+1} - x_{k+1}).$$

By writing $x_{k+1} - x_k = (x_{k+1} - z_{k+1}) + (z_{k+1} - x_k)$ and expanding the squared-norm of this term, and then combining the terms that are in the
inner product with $z_{k+1} - x_{k+1}$, we obtain
\[
\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - \|x_{k+1} - z_{k+1}\|^2 - \|z_{k+1} - x_k\|^2 \\
+ 2(x_{k+1} - z_{k+1})^T(x_k - \tau F(z_{k+1}) - z_{k+1}).
\]

We can further write [by adding and subtracting $\tau F(x_k)$]
\[
(x_{k+1} - z_{k+1})^T (x_k - \tau F(z_{k+1}) - z_{k+1}) = (x_{k+1} - z_{k+1})^T(x_k - \tau F(z_k) - z_{k+1}) \\
+ \tau(x_{k+1} - z_{k+1})^T(F(x_k) - F(z_{k+1}))
\]

Since $x_{k+1} \in K$ and $z_{k+1} = \Pi_K[x_k - \tau F(x_k)]$, the first term on the right hand side is nonnegative (by projection property). Thus, by using this and Lipschitz continuity of $F$, we have
\[
(x_{k+1} - z_{k+1})^T (x_k - \tau F(z_{k+1}) - z_{k+1}) \\
\leq \tau(x_{k+1} - z_{k+1})^T(F(x_k) - F(z_{k+1})) \\
\leq \tau L \|x_{k+1} - z_{k+1}\| \cdot \|x_k - z_{k+1}\| \\
\leq \frac{1}{2} \left(\|x_{k+1} - z_{k+1}\|^2 + \tau^2 L^2 \|x_k - z_{k+1}\|^2\right)
\]
By substituting the preceding estimate in (5), we obtain

\[
\begin{align*}
\|x_{k+1} - x^*\|^2 & \leq \|x_k - x^*\|^2 - \|x_{k+1} - z_{k+1}\|^2 - \|z_{k+1} - x_k\|^2 \\
& \quad + \|x_{k+1} - z_{k+1}\|^2 + \tau^2 L^2 \|x_k - z_{k+1}\| \\
& = \|x_k - x^*\|^2 - (1 - \tau^2 L^2) \|z_{k+1} - x_k\|^2.
\end{align*}
\]
Convergence of Extra-Gradient Method

Theorem 4 Let $F$ be monotone and Lipschitz continuous over $K$ with constant $L$. Let $SOL(K, F)$ be nonempty. Then, with $\tau < \frac{1}{L}$, the sequence $\{x_k\}$ generated by the extra-gradient method converges to a solution of $VI(K, F)$.

Proof The line of proof relies on the basic iterate relation, and follows a line of analysis similar to that of Theorem for co-coercive map and projection method. See FP-II Theorem 12.1.11.

For the estimate in basic relation to result in convergence, we need $\tau < \frac{1}{L}$.

- In practice, often $L$ is not available
- We can use diminishing step $\tau_k$ at iteration $k$, with $\sum_k \tau_k = +\infty$
- But the convergence will slow down
- We next consider a modification of the method not relying on the Lipschitz continuity
Hyperplane Projection Method

Like the extra-gradient method, this method generates an extra-iterate
\[ y_k = \Pi_K [x_k - \tau F(x_k)] \]. However, the use of this point in constructing the
new iterate is different.
In particular, Armijo search rule is used to determine a point \( z_k \) defining a
hyperplane
\[ H_k = \{ x \in \mathbb{R}^n \mid F(z_k)^T(x - z) = 0 \} \],
which separates \( x^k \) strongly from the solution set \( SOL(K, F) \) [for contin-
uous monotone map, this set is closed and convex - possibly empty].
In particular, \( z_k \) is such that, for some positive scalars \( t, \sigma, \tau \),
\[ F(z_k)^T(x_k - z_k) \geq \frac{t_k \sigma}{\tau} \|y_k - x_k\|^2 \],
which strongly separates \( x_k \) from \( SOL(K, F) \) whenever \( y_k \neq x_k \) in view of
\[ 0 \geq F(x^*)^T(x^* - z_k) \geq F(z_k)^T(x^* - z_k) \quad \text{ for all } x^* \in SOL(K, F). \]
The algorithm is initialized with the following parameters: 

\( k = 0, \ x_0 \in K, \ \tau > 0, \ \text{and} \ \sigma \in (0, 1) \). The algorithm proceeds as follows.

**Step 1:** Compute 

\[ y_k = \Pi_{K,D}[x_k - D^{-1}F(x_k)] \]

If \( y_k = x_k \), stop; we have a solution. Else, set \( t = 1 \) and go to Step 2.

**Step 2:** (Separation Test) If 

\[ F(ty_k + (1 - t)x_k)^T(x_k - y_k) \geq \sigma \tau \|x_k - y_k\|^2, \]

set \( t_k = t \) and go to Step 3; otherwise set \( t = \frac{t}{2} \) and repeat the test.

**Step 3:** Set 

\[ z_k = t_ky_k + (1 - t_k)x_k, \]

and compute 

\[ w_k = \Pi_{H_k}[x_k] = x_k - \frac{F(z_k)^T(x_k - z_k)}{\|F(z_k)\|^2} F(z_k). \]

**Step 4:** Set \( x_{k+1} = \Pi_K[w_k] \) and set \( k := k + 1 \), and go to Step 1.
Preliminary Results

Convergence proof uses three preliminary results.

**Projection properties** (Lemma 12.1.13)

(a) For any \( x, y \in \mathbb{R}^n \), we have

\[
\|\Pi_K [x] - \Pi_K [y]\|^2 \leq \|x - y\|^2 - \|\Pi_K [x] - x + y - \Pi_K [y]\|^2.
\]

(b) For any \( x \in K \) and \( y \in \mathbb{R}^n \), we have

\[
(x - y)^T (x - \Pi_K [y]) \geq \|x - \Pi_K [y]\|^2.
\]
Finite Termination of Loop in Step 2
To have the algorithm well defined, we need to verify that the Step 2 is exited after finitely many trials (test evaluations). The following lemma provides the guarantee.

**Lemma 2 (Based on Lemma 12.1.14)** Let $F$ be continuous over $K$, and let $x_k$ be such that $x_k \notin SOL(K, F)$. We then have

1. Step 2 is exited after finitely many trials.
2. For $z_k$ defined at Step 3, we have

$$F(z_k)^T(x_k - z_k) > \frac{t_k \sigma}{\tau} \|x_k - y_k\|^2 > 0,$$

**Proof** Suppose that Step 2 is never exited. Then, it follows that for all $i$,

$$F(2^{-i}y_k + (1 - 2^{-i})x_k)^T(x_k - y_k) < \frac{\sigma}{\tau} \|x_k - y_k\|^2.$$  

Letting $i \to \infty$, by continuity of $F$ we obtain

$$F(x_k)^T(x_k - y_k) < \frac{\sigma}{\tau} \|x_k - y_k\|^2.$$
Therefore, since $\tau > 0$,

$$\sigma \|x_k - y_k\|^2 \geq \tau F(x_k)^T(x_k - y_k) = (x_k - (x_k - \tau F(x_k)))^T(x_k - y_k) \geq \|x_k - y_k\|^2,$$

where the last inequality follows from $y_k = \prod_K [x_k - \tau F(x_k)]$, $x_k \in K$, and the projection property (second bullet).

Since $z_k = t_k y_k + (1 - t_k) x_k$, we have $x_k - z_k = t_k (x_k - y_k)$. Hence,

$$F(z_k)^T(x_k - z_k) = t_k F(z_k)^T(x_k - y_k).$$

By the construction of $z_k$ at Step 3 and from Step 2, we have

$$F(z_k)^T(x_k - y_k) > \sigma \tau \|x_k - y_k\|^2.$$ 

Combining the preceding two relations, we obtain

$$F(z_k)^T(x_k - z_k) > \frac{t_k \sigma}{\tau} \|x_k - y_k\|^2 > 0,$$
Auxiliary Lemma

Lemma 3 Let $F$ be continuous and monotone on $K$. Let $\{x_k\}$ be generated by the Hyperplane Projection Method. We then have

(a) The sequence $\{|x_k - x^*|\}$ is nonincreasing (hence, convergent) for every $x^* \in \text{SOL}(K, F)$.
(b) The sequence $\{x_k\}$ is bounded.
(c) If $\{x_k\}$ has an accumulation point in $\text{SOL}(K, F)$, then the whole sequence converges to this solution.
(d) $\lim_{k \to \infty} F(z_k)^T(x_k - z_k) = 0$.

Proof Note that (a) implies (b). Also, note that (a) and (b) together imply (c). Thus, only (a) and (d) need a proof.

Define the half-space

$$H_k^- = \{x \in \mathbb{R}^n \mid F(z_k)^T(x - z_k) \leq 0\}. $$
As noted earlier, by monotonicity of $F$, it can be seen that

$$\text{SOL}(K, F) \subseteq H_k^-.$$ 

Note that $x_k \notin H_k^-$ in view of preceding lemma [part (b)]. Therefore,

$$x^* = \Pi_K[x^*], \quad x^* = \Pi_{H_k^-}[x^*] \quad \text{for all } x^* \in \text{SOL}(K, F).$$

Furthermore, note that by definition of $w_k$, we have $w_k = \Pi_{H_k^-}[x_k]$. 

We now consider $x_{k+1} - x^*$ for arbitrary $x^* \in \text{SOL}(K, F)$. By projection property (first bullet), and the preceding relations, we have

$$\|x_{k+1} - x^*\|^2 = \|\Pi_K[w_k] - \Pi_K[x^*]\|^2$$

$$\leq \|w_k - x^*\|^2 - \|\Pi_K[w_k] - w_k\|^2$$

$$= \|\Pi_{H_k^-}[x_k] - \Pi_{H_k^-}[x^*]\|^2 - \|\Pi_K[w_k] - w_k\|^2$$

$$\leq \|x_k - x^*\|^2 - \|w_k - x^*\|^2 - \|\Pi_K[w_k] - w_k\|^2 \tag{7}$$
Thus, we have \( \|x_{k+1} - x^*\| \leq \|x_k - x^*\| \), showing that the scalar sequence \( \{\|x_k - x^*\|\} \) is nonincreasing, and hence convergent. This proves part (a). From (7), it also follows
\[
\|w_k - x_k\|^2 \leq \|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2.
\]
Since the scalar sequence \( \{\|x_k - x^*\|\} \) is convergent, we have
\[
\lim_{k \to \infty} \|w_k - x_k\| = 0.
\]
By the definition of \( w_k \), from the preceding relation we see that
\[
\lim_{k \to \infty} \frac{F(z_k)^T(x_k - z_k)}{\|F(z_k)\|} = 0.
\] (8)
Since \( \{x_k\} \) is bounded, \( \{F(x_k)\} \) is also bounded by continuity of \( F \). By \( y_k = \Pi_K [x_k - \tau F(x_k)] \), we obtain that \( \{y_k\} \) is also bounded. Since \( z_k \) is a convex combination of \( y_k \) and \( x_k \), we see that \( \{z_k\} \) is bounded, which by continuity of \( F \) implies that \( \{F(z_k)\} \) is bounded. Therefore, in view of (8),
\[
\lim_{k \to \infty} F(z_k)^T(x_k - z_k) = 0,
\]
showing part (d).
Theorem 5 Let $F$ be a continuous monotone mapping on $K$, and assume that $SOL(K, F)$ is nonempty. Then, the sequence $\{x_k\}$ generated by Hyperplane Projection Method converges to a solution $x^* \in SOL(K, F)$.

Proof Idea
The key is to show that a subsequence of $\{x_k\}$ converges to some point in $SOL(K, F)$, and then use the preceding lemma to conclude that the entire sequence converges to that point.

Showing that a subsequence of $\{x_k\}$ converges to a point in $SOL(K, F)$ is done by examining the following two possibilities for the stepsize sequence:

1. $\limsup_{k \to \infty} t_k > 0$

2. $\limsup_{k \to \infty} t_k = 0$, which implies $\lim_{k \to \infty} t_k = 0$. 