Lecture 18

Equation-based Algorithms for CPs

Regularity and Extensions

November 16, 2008

Outline

- Picking an element from $\partial \mathbf{F}_{\rm FB}$
- Pointwise FB-Regularity
- Nonsingularity of Newton approximation
- Constrained methods
- Extensions to mixed-CPs and box VIs

Obtaining an $H \in Jac\mathbf{F}_{\mathbf{FB}}(x)$

1. Set $\beta \equiv \{i : x_i = 0 = F_i(x)\}$.

2. Choose $z \in \mathbb{R}^n$ with $z_i \neq 0, \quad \forall i \in \beta$

3. For each $i \notin \beta$, set the ith column of H^T as
$$\left(\frac{x_i}{\sqrt{x_i^2 + F_i(x)^2}} - 1\right) e^i + \left(\frac{F_i(x)}{\sqrt{x_i^2 + F_i^2}} - 1\right) \nabla F_i(x).$$

4. For each $i \in \beta$ set the ith column of H^T equal to
$$\left(\frac{z_i}{\sqrt{z_i^2 + (\nabla F_i(x)^T z)^2}} - 1\right) e^i + \left(\frac{\nabla F_i(x)^T z}{\sqrt{z_i^2 + (\nabla F_i(x)^T z)}} - 1\right) \nabla F_i(x).$$

Proof of Correctness

Proposition 1 *The matrix H calculated by the above procedure is an element of Jac* $\mathbf{F}_{FB}$

Proof:

- Suffices to construct a sequence $\{y^k\}$ converging to x such that $\mathbf{F}_{\mathrm{FB}}$ is F-differentiable at each y^k and $\{J\mathbf{F}_{\mathrm{FB}}(y^k)\}$ converges to H.
- Let $y^k \equiv x + \epsilon_k z$ where z is from (2.) and $\{\epsilon_k\} \to 0$.
- If $i \notin \beta$, then either x_i or $F_i(x)$ are nonzero for all. In addition, $z_i \neq 0$ for all $i \in \beta$. Thus, we may assume that ϵ_k is small enough that either $y_i^k \neq 0$ or $F_i(y^k) \neq 0$; Therefore $\mathbf{F}_{\mathrm{FB}}$ is F-differentiable at y^k
- Clearly, if $i \notin \beta$, the ith row of $J\mathbf{F}_{\mathrm{FB}}(y^k)$ tends to the ith row of H. It suffices to consider only those rows that belong to β. For such indices, the ith row of $J\mathbf{F}_{\mathrm{FB}}(y^k)$ is given by

$$(a_i(y^k) - 1)(e^i)^T + (b_i(y^k) - 1)\nabla F_i(y^k)^T,$$

  where

$$a_i(y^k) \equiv \frac{\epsilon_k z_i}{\sqrt{\epsilon_k^2 z_i^2 + F_i(y^k)^2}} \text{ and } b_i(y^k) \equiv \frac{F_i(y^k)}{\sqrt{\epsilon_k^2 z_i^2 + F_i(y^k)^2}}.$$

- By a Taylor's series expansion around x we have for each $i \in \beta$,

$$F_i(y^k) = F_i(x) + \epsilon_k \nabla F_i(\zeta^k)^T z = \epsilon_k \nabla F_i(\zeta^k)^T z,$$

for some ζ^k on segment joining y^k and x. Hence

$$(a_i(y^k), b_i(y^k)) = \frac{(z_i, \nabla F_i(\zeta^k)^T z)}{\sqrt{z_i^2 + (\nabla F_i(\zeta^k)^T z)^2}}.$$

- Clearly $\{\zeta^k\} \to x$, implying that $\lim_{k\to\infty} J\mathbf{F}_{\mathrm{FB}}(y^k)$, given continuity of JF, gives us

$$\left(\frac{z_i}{\sqrt{z_i^2 + (\nabla F_i(x)^T z)^2}} - 1\right) e^i + \left(\frac{\nabla F_i(x)^T z}{\sqrt{z_i^2 + (\nabla F_i(x)^T z)}} - 1\right) \nabla F_i(x).$$

Pointwise FB regularity

When is a stationary point of $\theta_{\rm FB}$ a solution to NCP(F)?

- Recall that $\nabla\theta_{\rm FB} = H^T\mathbf{F}_{\rm FB}$, where $H \in \partial\mathbf{F}_{\rm FB}$.
- Therefore if $\nabla\theta_{\rm FB}(x) = 0$, then $\mathbf{F}_{\rm FB}(x) = 0$ if $\partial\mathbf{F}_{\rm FB}$ contains a nonsingular matrix
- The discussion in this subsection pertains to settings when the generalized Jacobian of $\mathbf{F}_{\rm FB}$ contains only singular matrices; such points are called singular stationary points of $\theta_{\rm FB}$
- For this purpose, we introduce the notion of **FB regularity**
- Recall that
$$\nabla\theta_{\rm FB} = D_a\mathbf{F}_{\rm FB}(x) + JF^TD_b\mathbf{F}_{\rm FB},$$
where the dependence on x is suppressed

- The signs of the components of $D_a\mathbf{F}_{\rm FB}$ and $D_b\mathbf{F}_{\rm FB}$ are of relevance and we introduce several index sets*:
  - $\mathcal{C} \equiv \{i : x_i \geq 0, F_i(x) \geq 0, x_iF_i(x) = 0.\}$ (complementary indices)
  - $\mathcal{R} \equiv \{1,\ldots,n\}\backslash\mathcal{C}$ (residual indices)
  - $\mathcal{P} \equiv \{i \in \mathcal{R} : x_i > 0, F_i(x) > 0\}$ (positive indices)
  - $\mathcal{N} = \mathcal{R}\backslash\mathcal{P}.$ (negative indices)

*Note that the dependence on x is suppressed

Recapping the structure of the diagonal matrices

- The diagonal mtrices D_a and D_b belong to $\mathcal{D}_a$ and $\mathcal{D}_b$ with diagonal elements a_i and b_i, respectively.

- The definitions of (a_i, b_i) are given by

$$(a_i(x), b_i(x)) \begin{cases} \equiv \frac{(x_i, F_i(x))}{\sqrt{x_i^2 + F_i(x)^2}} - (1,1), & \text{if } (x_i, F_i(x)) \neq 0 \\ \in \text{ cl } \mathbf{B}(0,1) - (1,1), & \text{if } (x_i, F_i(x)) = 0. \end{cases}$$

- Therefore a_i and b_i are of the form: $a_i(x) \equiv \xi_i - 1$ and $b_i(x) = \rho_i - 1$ where

$$\xi_i^2 + \rho_i^2 \begin{cases} = 1, & (x_i, F_i(x)) = 0 \\ \leq 1, & (x_i, F_i(x)) \neq 0, \quad (\xi_i, \rho_i) \in cl(B(0,1)). \end{cases}$$

- Let $v \equiv D_a \mathbf{F}_{\rm FB}$ and $z \equiv D_b \mathbf{F}_{\rm FB}$. Then

$$z_i > 0 \implies v_i > 0 \implies \theta_{\rm FB}(x_i, F_i(x)) < 0,$$
$$z_i = 0 \implies v_i = 0 \implies \theta_{\rm FB}(x_i, F_i(x)) = 0,$$
$$z_i < 0 \implies v_i < 0 \implies \theta_{\rm FB}(x_i, F_i(x)) > 0.$$

- Furthermore

$$(D_a\mathbf{F}_{\rm FB})_i > 0 \Leftrightarrow (D_b\mathbf{F}_{\rm FB})_i > 0 \Leftrightarrow i \in \mathcal{P},$$
$$(D_a\mathbf{F}_{\rm FB})_i = 0 \Leftrightarrow (D_b\mathbf{F}_{\rm FB})_i = 0 \Leftrightarrow i \in \mathcal{C},$$
$$(D_a\mathbf{F}_{\rm FB})_i < 0 \Leftrightarrow (D_b\mathbf{F}_{\rm FB})_i < 0 \Leftrightarrow i \in \mathcal{N}.$$

This leads to a definition of FB-regularity:

Definition 1 A point $x \in \mathbb{R}^n$ is called **FB-regular** if for every vector $z \neq 0$ such that

$$z_{\mathcal{C}} = 0, z_{\mathcal{P}} > 0, z_{\mathcal{N}} < 0,$$

there exists $0 \neq y \in \mathbb{R}^n$ such that $y_{\mathcal{C}} = 0, y_{\mathcal{P}} \geq 0, y_{\mathcal{N}} \leq 0$, and $z^T JF(x) y \geq 0$.

Stationarity of $\theta_{\rm FB}$, FB-regularity and SOL(F)

Theorem 1 Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. If $x \in \mathbb{R}^n$ is a stationary point of $\theta_{\rm FB}$, then x is a solution of NCP(F) if and only if x is an FB-regular point of $\theta_{\rm FB}$.

Proof:

($\Leftarrow$) Assume that x is a solution of NCP(F). Then, x is a global minimizer of $\theta_{\rm FB}(x)$ and therefore a stationary point of $\theta_{\rm FB}(x)$. Consequently $\mathcal{P} = \mathcal{N} = \emptyset$. Therefore $z = z_{\mathcal{C}}$ and FB-regularity holds vacuously since there is no nonzero vector satisfying $z_{\mathcal{C}} = 0, z_{\mathcal{P}} > 0, z_{\mathcal{N}} < 0$.

($\Rightarrow$) Conversely, let x be FB-regular such that $\nabla \theta_{\rm FB}(x) = 0$. The stationarity condition may then be written as

$$D_a \mathbf{F}_{\rm FB} + JF(x)^T D_b \mathbf{F}_{\rm FB} = 0$$

implying that

$$y^T D_a \mathbf{F}_{\rm FB} + y^T JF(x)^T D_b \mathbf{F}_{\rm FB} = 0 \qquad (*)$$

for any $y \in \mathbb{R}^n$.

- Let us proceed by contradiction; suppose x is not a solution to NCP(F). Then, the set $\mathcal{R}$ is nonempty (some indices are non-complementary).[†] Then $i \in \mathcal{R}$ implies that $z \equiv D_b \mathbf{F}_{\rm FB}$ is a nonzero vector by noting that

$$(D_a \mathbf{F}_{\rm FB})_i = 0 \Leftrightarrow (D_b \mathbf{F}_{\rm FB})_i = 0 \Leftrightarrow i \in \mathcal{C}.$$

- Moreover, by FB-regularity, we have that $z_{\mathcal{C}} = 0, z_{\mathcal{P}} > 0, z_{\mathcal{N}} < 0$. Recall that the components of $v \equiv D_a \mathbf{F}_{\rm FB}$ and $z = D_b \mathbf{F}_{\rm FB}$ have the same signs componentwise, we have that

$$y^T (D_a \mathbf{F}_{\rm FB}) = y_{\mathcal{C}}^T (D_a \mathbf{F}_{\rm FB})_{\mathcal{C}} + y_{\mathcal{P}}^T (D_a \mathbf{F}_{\rm FB})_{\mathcal{P}} + y_{\mathcal{N}}^T (D_a \mathbf{F}_{\rm FB})_{\mathcal{N}} > 0,$$

since $y_{\mathcal{R}} \neq 0$ and $y_{\mathcal{C}} = 0, y_{\mathcal{P}} \geq 0, y_{\mathcal{N}} \leq 0$.

- Furthermore, $y^T JF(x)^T (D_b \mathbf{F}_{\rm FB}) = y^T JF(x)^T z \geq 0$, where the last inequality follows from FB-regularity.

- Together, these contradict (*) and therefore $\mathcal{R} = \emptyset$ and x is a solution of NCP(F).

[†] Note that $x \in SOL(F)$ if and only if $\mathcal{R} = \emptyset$.

Comments

- If $JF(x) \succeq 0$ at x, then the earlier definition may be employed to show that x is FB-regular (set $y = z$).

- In fact, $z^T JF(x)y \geq 0$ is equivalent to
$$\sum_{i \in \mathcal{P}} z_i (JF(x)y)_i + \sum_{i \in \mathcal{N}} z_i (JF(x)y)_i.$$
By noting that $z_{\mathcal{P}} > 0$ and $z_{\mathcal{N}} < 0$, we have that this inequality holds if
$$\nabla F_i(x)^T y \geq 0, \quad \forall i \in \mathcal{P}$$
$$\nabla F_i(x)^T y \leq 0, \quad \forall i \in \mathcal{N}.$$

- Therefore a sufficient condition for x to be an FB-regular point is that there exist a $y \neq 0$ with $y_{\mathcal{C}} = 0, y_{\mathcal{P}} \geq 0, y_{\mathcal{N}} \leq 0$ along with the above two inequalities.

- This in turn is equivalent to the existence of a nonzero $(u_{\mathcal{P}}, u_{\mathcal{N}})$ such that
$$\begin{pmatrix} J_{\mathcal{P}} F_{\mathcal{P}} & -J_{\mathcal{P}} F_{\mathcal{P}} \\ -J_{\mathcal{P}} F_{\mathcal{N}} & J_{\mathcal{N}} F_{\mathcal{N}} \end{pmatrix} \begin{pmatrix} u_{\mathcal{P}} \\ u_{\mathcal{N}} \end{pmatrix} \geq 0, u_{\mathcal{P}}, u_{\mathcal{N}} \geq 0.$$

- Recall that for a matrix $M \in \mathbb{R}^{n\times n}$ for which there exists a u satisfying

$$Mu \geq 0$$
$$u \geq 0,$$

is called an S_0 matrix (containing the class of S-matrices or Stieltje matrices.)

- **Advantage:** Computing whether this sufficient condition does indeed hold is a finite procedure and requires solving an LP while verifying FB-regularity cannot in general be shown using a finite procedure.

- To relate the matrix

$$M = \begin{pmatrix} J_{\mathcal{P}}F_{\mathcal{P}} & -J_{\mathcal{P}}F_{\mathcal{P}} \\ -J_{\mathcal{P}}F_{\mathcal{N}} & J_{\mathcal{N}}F_{\mathcal{N}} \end{pmatrix}$$

to the Jacobian $JF(x)$, we use a sign matrix $\Lambda(x)$, a diagonal matrix whose diagonal entries λ_i are defined as

$$\lambda_i = \begin{cases} 1, & i \in \mathcal{P} \\ -1, & i \in \mathcal{N} \\ 0, & i \in \mathcal{C}. \end{cases}$$

It follows that M is a principal submatrix of $\Theta(x) = \Lambda(x) JF(x) \Lambda(x)$ with rows and colums corresponding to $\mathcal{R} = \mathcal{P} \cup \mathcal{N}$ of the vector x.

- This leads to the following definition:

  Definition 2 Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable. We say that $JF(x)$ is a signed S_0 matrix if $\Theta(x)_{\mathcal{RR}}$ is an S_0 matrix. Furthermore, we say that F has **differentiable signed S_0** property at x if $JF(x)$ is a signed S_0 matrix. If F has the differentiable signed S_0 property at every point in its domain, then F is a signed S_0 function.

- Therefore the signed S_0 property of JF(x) is a sufficient condition for x to be FB-regular. Furthermore, even singular stationary points of $\theta_{\rm FB}$ can be shown to be solutions to the NCP as the following corollary specifies.

  Corollary 1 Suppose that F is continuously differentiable. If F has the differentiable signed S_0 property at every singular stationary point, then every stationary point of $\theta_{\rm FB}$ is a solution to NCP(F). In this case, every accumulation point of the sequence of iterates produced by the (FBLSA) solves the NCP.

Nonsingularity of Newton Approximation

- the linear Newton approximation, denoted by T, should be chosen such that

$$T(x) \subseteq \mathcal{D}_a(x) + \mathcal{D}_b(x)JF(x).$$

- What **sufficient** conditions do we need to ensure that all matrices in $\mathcal{D}_a(x) + \mathcal{D}_b(x)JF(x)$, we have nonsingularity.

- While nonsingularity doesnt guarantee that the sufficient descent criterion will be met, it does allow for **fast local convergence**(see Th. 7.5.15)

- To ensure that $\mathcal{D}_a(x) + \mathcal{D}_b(x)JF(x)$ contains only nonsingular matrices, additional properties must be imposed on F.

Lemma 1 Let $M \in \mathbb{R}^{n\times n}$ be a given matrix. The following two statements are equivalent:

(a) M is a P_0 matrix

(b) Every matrix of the form $D_a + D_b M$ is nonsingular for all nonnegative(nonpositive) diagonal matrices D_a and D_b with $(D_a)_{ii} > (<) 0$ for all $i = 1, \ldots, n$.

Theorem 2 Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable in a neighborhood of $x \in \mathbb{R}^n$. Let $M \equiv JF(x)$; also let $\bar{\alpha} \equiv \gamma \cup \beta \cup \delta$ be the complement if α in $\{1, \ldots, n\}$, where

$$\alpha \equiv \{i : x_i = 0 < F_i(x)\}$$
$$\beta \equiv \{i : x_i = 0 = F_i(x)\}$$
$$\gamma \equiv \{i : x_i > 0 = F_i(x)\}$$
$$\delta \equiv \{1, \ldots, n\} \backslash (\alpha \cup \beta \cup \gamma).$$

Assume that

(a) the submatrices $M_{\tilde{\gamma}\tilde{\gamma}}$ are nonsingular for all $\tilde{\gamma}$ satisfying $\gamma \subset \tilde{\gamma} \subset \gamma \cup \beta$.

(b) the Schur complement of $M_{\gamma\gamma}$ in $M_{\bar{\alpha}\bar{\alpha}}$ is a P_0 matrix

Constrained Methods

- Consider a constrained formulation of the NCP: $\begin{cases} (CNCP)\ \text{min} & \theta_{\mathbf{FB}}(x) \\ \textbf{subject to} & x \in \mathbb{R}^n_+, \end{cases}$

- Suppose that F is defined over $\mathbb{R}^n_+$ in (CNCP). Specifically, we consider algorithms that maintain nonnegativity of the iterates (for instance, F may lose differentiability outside $\mathbb{R}^n_+$).

- Note that x is a stationary point of (CNCP) if $0 \le x \perp \nabla\theta_{\mathrm{FB}}(x) \ge 0$ or $\min(x, \nabla\theta_{\mathrm{FB}}(x)) = 0$.

Theorem 3 *Suppose that $F : \Omega \subset \mathbb{R}^n_+ \to \mathbb{R}^n$ be continuously differentiable on the open set Ω and that x is a constrained stationary point of (CNCP). Then x is a stationary point of NCP(F) if and only if x is a FB-regular point of θ_{FB}*

Proof:

- One direction (forward) is the same as the earlier theorem proved.

- We prove the other direction. Suppose that x is an FB-regular constrained stationary point but not a solution of NCP(F). Let I_0 and I_+ represent the following:

$$I_0 \equiv \{i : x_i = 0\}$$
$$I_+ \equiv \{i : x_i > 0\}.$$

- The stationarity conditions may then be written as

$$(\nabla \theta_{\rm FB}(x))_i = 0, i \in I_+$$
$$(\nabla \theta_{\rm FB}(x))_i \geq 0, i \in I_0.$$

- Let $z = D_b \mathbf{F}_{\rm FB}(x)$ where D_b is an arbitrary diagonal matrix in $\mathcal{D}_b(x)$. Since $x \notin SOL(F)$, $z \neq 0$. Thenn let y be the vector in the definition of FB-regularity. Then

$$i \in I_0 \implies x_i = 0 \implies i \in \mathcal{C} \cup \mathcal{N} \implies y_i \leq 0.$$

(note that $i \in \mathcal{N}$ otherwise $x \in SOL(F)$)

- Therefore,

$$y^T \nabla \theta_{\rm FB}(x) = y_{I_0}^T \nabla_{I_0} \theta_{\rm FB}(x) \leq 0. \qquad (**)$$

But from the definition, we have $y^T JF(x)^T z \geq 0$. Moreover $y^T D_a \mathbf{F}_{\rm FB}(x) > 0$ as in the earlier proof. But these two inequalities contradict (**). This concludes the proof.

Algorithms

- Next, we discuss some algorithms for solving such problems
- Given an iterate x^k, we compute the next iterate by solving LCP$(q^k, JF(x^k))$, where $q^k \equiv F(x^k) - JF(x^k)x^k$.
- Globalization of this algorithm is achieved by using the FB merit function along which we conduct a linesearch using the steepest descent direction.
- Such a direction is computed by solving the problem

$$\begin{cases} \min & \nabla \theta_{\rm FB}(x^k)^T d + \frac{1}{2} d^T d \\ \text{subject to} & x^k + d \in \mathbb{R}^n_+. \end{cases}.$$

- It may be shown that

$$\widehat{d}^k = \max(-x^k, -\nabla \theta_{\rm FB}(x^k)) = -\min(x^k, \theta_{\rm FB}(x^k)).$$

Moreover $\widehat{d}^k = 0 \Leftrightarrow \min(x^k, \nabla \theta_{\rm FB}(x^k)) = 0$. Thus if x^k is not a stationary point of (CNCP), then $\widehat{d}^k$ is a nonzero descent direction of $\theta_{\rm FB}$ at x^k,

$$\nabla \theta_{\rm FB}(x^k)^T \widehat{d}^k + \tfrac{1}{2}(\widehat{d}^k)^T \widehat{d}^k) < 0 \text{ implying that } \|\widehat{d}^k\| < \sqrt{-2\nabla \theta_{\rm FB}(x^k)^T \widehat{d}^k}.$$

- Note that this direction is a fall-back direction.

- Constrained FB Linesearch Algorithm

0. $x^0, \rho > 0, p > 1, \gamma \in (0,1)$
1. k $= 0$
2. If x^k is a stationary point of (CNCP), stop.
3. Find a solution y^{k+1} of the LCP$(q^k, JF(x^k))$ and set $d^k \equiv y^{k+1} - x^k$. If the LCP is not solvable or if $\nabla\theta_{\rm FB}(x^k)^T d^k \le -\rho\|d\|^p$, is not satisfied, set $d^k \equiv \widehat{d}^k$.
4. Find the smallest nonnegative integer i_k such that with $i = i_k$,
$$\theta_{\rm FB}(x^k + 2^{-i}d^k) \le \theta_{\rm FB}(x^k) - \min\{-2^{-i}\gamma\nabla\theta_{\rm FB}(x^k)^T d^k, (1-\gamma)\theta_{\rm FB}(x^k)\}.$$
5. Set $x^{k+1} \equiv x^k + \tau_k d^k$ and $k \equiv k+1$; go to 2.

Extensions

- Discuss extensions to NCPs
- Finite lower (or upper) bounds
- Mixed-complementarity problems
- Box-constrained VIs

Finite lower(or upper) bounds

- Consider the VI(K,F) in which the set K is defined a
$$K \equiv \{x \in \mathbb{R}^n : x \geq a\},$$
where a is a given vector.

- One option is to define $y = x - a$ and convert the problem into an NCP. Alternate approach requires observing that
$$(x \in SOL(K,F)) \Leftrightarrow [x_i = a_i \implies F_i(x) \geq 0] \text{ and } [x_i > a_i \implies F_i(x) = 0].$$

- An equation reformulation for this VI is given by
$$0 = \mathbf{F}^{\ell}_{\mathrm{FB}}(x) \equiv \begin{pmatrix} \theta_{\mathrm{FB}}(x_1 - a_1, F_1(x)) \\ \vdots \\ \theta_{\mathrm{FB}}(x_n - a_n, F_n(x)) \end{pmatrix},$$
which immediately suggests the merit function given by $\theta^{\ell}_{\mathrm{FB}}(x) = \frac{1}{2}(\mathbf{F}^{\ell}_{\mathrm{FB}}(x))^T \mathbf{F}^{\ell}_{\mathrm{FB}}(x)$.

- Similarly for upper bounds.

Mixed-complementarity Problems

- Consider the MiCP(G,H) given by

$$G(u,v) = 0, \qquad u \text{ free}$$
$$0 \leq v \perp H(u,v) \geq 0,$$

where $G :\Rightarrow \mathbb{R}^n \to \mathbb{R}^{n_1}$ and $H : \mathbb{R}^n \Rightarrow \mathbb{R}^{n_1}$ such that $n_1 + n_2 = n$.

- The FB-based equation formulation is given by

$$0 = \mathbf{F}_{\mathrm{FB}}^{\mathrm{MCP}}(u,v) \equiv \begin{pmatrix} G(u,v) \\ \theta_{\mathrm{FB}}(v_1, H_1(u,v)) \\ \vdots \\ \theta_{\mathrm{FB}}(v_n, H_n(u,v)) \end{pmatrix}.$$

- The associated merit function is given by $\theta_{\mathrm{FB}}^{\mathrm{MCP}}(u,v) = \frac{1}{2}\mathbf{F}_{\mathrm{FB}}^{\mathrm{MCP}}(u,v)^T\mathbf{F}_{\mathrm{FB}}^{\mathrm{MCP}}(u,v)$.

Key properties

Proposition 2 Let $G, H \in C^2$ be as defined earlier. Then the following two statements are valid:

1. $\mathbf{F}_{\mathrm{FB}}^{\mathrm{MCP}}$ is semismooth and θ_{FB} is continuously differentiable. If G and H have Lipschitz continuous derivatives, then $\mathbf{F}_{\mathrm{FB}}^{\mathrm{MCP}}$ is strongly semismooth.

2. If $g_{\mathrm{FB}}(a,b)$ is a linear Newton approximation of $\psi_{\mathrm{FB}}(a,b)$, then a linear Newton approximation scheme of $\mathbf{F}_{\mathrm{FB}}^{\mathrm{MCP}}$ is given by
$$\mathcal{A}(u,v) = \{\begin{pmatrix} J_u G & J_v G \\ D_b J_u H(u,v) & D_a + D_b J_v H(u,v), \end{pmatrix}\}$$
where D_a and D_b are $n_2 \times n_2$ diagonal matrices whose ith entries are the first and second elements of an element $\xi_i \in g_{\mathrm{FB}}(v_i, H_i(u,v))$. If $g_{\mathrm{FB}}(a,b) \subseteq \partial\psi_{\mathrm{FB}}(a,b)$, then
$$\nabla\theta_{\mathrm{FB}}^{\mathrm{MCP}}(u,v) = A^T \mathbf{F}_{\mathrm{FB}}^{\mathrm{MCP}}(u,v), \forall A \in \mathcal{A}(u,v).$$

Box-constrained VIs

- Consider the box-constrained VI(K,F) with

$$K \equiv \{x \in \mathbb{R}^n : a_i \le x \le b_i, i = 1, \ldots, n\}.$$

- Just as in the 1-sided case, we have

$$\begin{aligned} x_i = a_i &\implies F_i \ge 0 \\ a_i < x_i < b_i &\implies F_i = 0 \\ x_i = b_i &\implies F_i \le 0. \end{aligned}$$

- Can we define a generalization of a C-function that captures the above properties. Specifically, given a pair of scalars $-\infty \le \tau < \tau' \le \infty$, we call a function $\phi(\tau', \tau, ., .) : \mathbb{R}^2 \to \mathbb{R}$ a B-function if

$$\phi(\tau', \tau, r, s) = 0 \Leftrightarrow \tau \le r \le \tau'$$

and (r,s) satisfies:

$$\begin{aligned} r = \tau &\implies s \ge 0 \\ \tau' > r > \tau &\implies s = 0 \\ r = \tau' &\implies s \le 0. \end{aligned}$$

- Note that a B-function with $\tau = 0, \tau = \infty$ is a C-function.
- Conversely given an C-function that satisfies the sign reversal property given by

$$\psi(a,b) \le 0 \implies ab \le 0$$

and any pair of scalars $-\infty \le \tau < \tau' \le \infty$, the function $\phi(\tau, \tau'; ., .)$ is defined by

$$\phi(\tau,\tau';r,s) = \begin{cases} \psi(\tau'-r,-s) & -\infty = \tau < \tau' < \infty \\ \psi(r-\tau,\psi(\tau'-r,-s)) & -\infty < \tau < \tau' < \infty \\ \psi(r-\tau,s) & -\infty = \tau < \tau' = \infty \\ s & -\infty = \tau < \tau' = \infty. \end{cases}$$

Proof:

- When one bound is finite, define a new variable (as shown earlier)
- Need to verify when both bounds are finite.
- In this case, we have

$$\phi(\tau,\tau';r,s) \Leftrightarrow 0 \le r - \tau \perp \psi(\tau'-r,-s) \ge 0.$$

- If $\tau < r$, then $\psi(\tau'-r,-s) = 0$ which implies that

$$0 \le \tau' - r \perp s \le 0.$$

- If $\tau' > r$, implies that $s = 0$. If $\tau' = r$, then $s \leq 0$.
- If $\tau = r$, then $\psi(\tau' - r, -s) \geq 0$ and $\tau' - r > 0$. Therefore by the sign-reversal property, $s \geq 0$ and ϕ is a B-function.