Outline

• Picking an element from $\partial F_{FB}$

• Pointwise FB-Regularity

• Nonsingularity of Newton approximation

• Constrained methods

• Extensions to mixed-CPs and box VIs
**Obtaining an** \( H \in \text{Jac}F_{FB}(x) \)

1. Set \( \beta \equiv \{i : x_i = 0 = F_i(x)\} \).

2. Choose \( z \in \mathbb{R}^n \) with \( z_i \neq 0, \quad \forall i \in \beta \)

3. For each \( i \notin \beta \), set the \( i \)th column of \( H^T \) as

\[
\left( \frac{x_i}{\sqrt{x_i^2 + F_i(x)^2}} - 1 \right) e^i + \left( \frac{F_i(x)}{\sqrt{x_i^2 + F_i^2}} - 1 \right) \nabla F_i(x).
\]

4. For each \( i \in \beta \) set the \( i \)th column of \( H^T \) equal to

\[
\left( \frac{z_i}{\sqrt{z_i^2 + (\nabla F_i(x)^T z)^2}} - 1 \right) e^i + \left( \frac{\nabla F_i(x)^T z}{\sqrt{z_i^2 + (\nabla F_i(x)^T z)^2}} - 1 \right) \nabla F_i(x).
\]
Proof of Correctness

Proposition 1 The matrix $H$ calculated by the above procedure is an element of $\text{Jac} \ F_{FB}$

Proof:

• Suffices to construct a sequence $\{y^k\}$ converging to $x$ such that $F_{FB}$ is $F$-differentiable at each $y^k$ and $\{JF_{FB}(y^k)\}$ converges to $H$.

• Let $y^k \equiv x + \epsilon_k z$ where $z$ is from (2.) and $\{\epsilon_k\} \to 0$.

• If $i \notin \beta$, then either $x_i$ or $F_i(x)$ are nonzero for all. In addition, $z_i \neq 0$ for all $i \in \beta$. Thus, we may assume that $\epsilon_k$ is small enough that either $y^k_i \neq 0$ or $F_i(y^k) \neq 0$; Therefore $F_{FB}$ is $F$-differentiable at $y^k$

• Clearly, if $i \notin \beta$, the $i$th row of $JF_{FB}(y^k)$ tends to the $i$th row of $H$. It suffices to consider only those rows that belong to $\beta$. For such indices, the $i$th row of $JF_{FB}(y^k)$ is given by

\[
(a_i(y^k) - 1)(e^i)^T + (b_i(y^k) - 1)\nabla F_i(y^k)^T,
\]

where

\[
a_i(y^k) \equiv \frac{\epsilon_k z_i}{\sqrt{\epsilon_k^2 z_i^2 + F_i(y^k)^2}} \quad \text{and} \quad b_i(y^k) \equiv \frac{F_i(y^k)}{\sqrt{\epsilon_k^2 z_i^2 + F_i(y^k)^2}}.
\]
By a Taylor’s series expansion around \( x \) we have for each \( i \in \beta \),

\[
F_i(y^k) = F_i(x) + \epsilon_k \nabla F_i(\zeta^k)^T z = \epsilon_k \nabla F_i(\zeta^k)^T z,
\]

for some \( \zeta^k \) on segment joining \( y^k \) and \( x \). Hence

\[
(a_i(y^k), b_i(y^k)) = \frac{(z_i, \nabla F_i(\zeta^k)^T z)}{\sqrt{z_i^2 + (\nabla F_i(\zeta^k)^T z)^2}}.
\]

Clearly \( \{\zeta^k\} \to x \), implying that \( \lim_{k \to \infty} JF_{FB}(y^k) \), given continuity of \( JF \), gives us

\[
\left( \frac{z_i}{\sqrt{z_i^2 + (\nabla F_i(x)^T z)^2}} - 1 \right) e^i + \left( \frac{\nabla F_i(x)^T z}{\sqrt{z_i^2 + (\nabla F_i(x)^T z)^2}} - 1 \right) \nabla F_i(x).
\]
Pointwise FB regularity

When is a stationary point of $\theta_{FB}$ a solution to $\text{NCP}(F)$?

- Recall that $\nabla \theta_{FB} = H^T F_{FB}$, where $H \in \partial F_{FB}$.

- Therefore if $\nabla \theta_{FB}(x) = 0$, then $F_{FB}(x) = 0$ if $\partial F_{FB}$ contains a nonsingular matrix.

- The discussion in this subsection pertains to settings when the generalized Jacobian of $F_{FB}$ contains only singular matrices; such points are called singular stationary points of $\theta_{FB}$.

- For this purpose, we introduce the notion of $\text{FB regularity}$.

- Recall that

\[
\nabla \theta_{FB} = D_a F_{FB}(x) + J F^T D_b F_{FB},
\]

where the dependence on $x$ is suppressed.
• The signs of the components of $D_a F_{FB}$ and $D_b F_{FB}$ are of relevance and we introduce several index sets*:

- $C \equiv \{ i : x_i \geq 0, F_i(x) \geq 0, x_i F_i(x) = 0. \}$ (complementary indices)
- $\mathcal{R} \equiv \{1, \ldots, n\} \setminus C$ (residual indices)
- $\mathcal{P} \equiv \{ i \in \mathcal{R} : x_i > 0, F_i(x) > 0 \}$ (positive indices)
- $\mathcal{N} = \mathcal{R} \setminus \mathcal{P}$. (negative indices)

*Note that the dependence on $x$ is suppressed
Recapping the structure of the diagonal matrices

- The diagonal matrices $D_a$ and $D_b$ belong to $D_a$ and $D_b$ with diagonal elements $a_i$ and $b_i$, respectively.

- The definitions of $(a_i, b_i)$ are given by

$$ (a_i(x), b_i(x)) \begin{cases} \equiv \frac{(x_i, F_i(x))}{\sqrt{x_i^2 + F_i(x)^2}} - (1, 1), & \text{if } (x_i, F_i(x)) \neq 0 \\ \in \text{cl } B(0, 1) - (1, 1), & \text{if } (x_i, F_i(x)) = 0. \end{cases} $$

- Therefore $a_i$ and $b_i$ are of the form: $a_i(x) \equiv \xi_i - 1$ and $b_i(x) = \rho_i - 1$ where

$$ \xi_i^2 + \rho_i^2 \begin{cases} = 1, & (x_i, F_i(x)) = 0 \\ \leq 1, & (x_i, F_i(x)) \neq 0, \quad (\xi_i, \rho_i) \in cl(B(0, 1)). \end{cases} $$

- Let $v \equiv D_a F_{FB}$ and $z \equiv D_b F_{FB}$. Then

$$ z_i > 0 \implies v_i > 0 \implies \theta_{FB}(x_i, F_i(x)) < 0, $$

$$ z_i = 0 \implies v_i = 0 \implies \theta_{FB}(x_i, F_i(x)) = 0, $$

$$ z_i < 0 \implies v_i < 0 \implies \theta_{FB}(x_i, F_i(x)) > 0. $$
Furthermore

\[(D_a F_{FB})_i > 0 \iff (D_b F_{FB})_i > 0 \iff i \in \mathcal{P},\]
\[(D_a F_{FB})_i = 0 \iff (D_b F_{FB})_i = 0 \iff i \in \mathcal{C},\]
\[(D_a F_{FB})_i < 0 \iff (D_b F_{FB})_i < 0 \iff i \in \mathcal{N}.\]

This leads to a definition of FB-regularity:

**Definition 1** A point \(x \in \mathbb{R}^n\) is called **FB-regular** if for every vector \(z \neq 0\) such that

\[z_{\mathcal{C}} = 0, z_{\mathcal{P}} > 0, z_{\mathcal{N}} < 0,\]

there exists \(0 \neq y \in \mathbb{R}^n\) such that \(y_{\mathcal{C}} = 0, y_{\mathcal{P}} \geq 0, y_{\mathcal{N}} \leq 0\), and \(z^T JF(x)y \geq 0\).
Stationarity of $\theta_{FB}$, FB-regularity and SOL(F)

**Theorem 1** Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. If $x \in \mathbb{R}^n$ is a stationary point of $\theta_{FB}$, then $x$ is a solution of NCP(F) if and only if $x$ is an FB-regular point of $\theta_{FB}$.

**Proof:**

($\Leftarrow$) Assume that $x$ is a solution of NCP(F). Then, $x$ is a global minimizer of $\theta_{FB}(x)$ and therefore a stationary point of $\theta_{FB}(x)$. Consequently $P = N = \emptyset$. Therefore $z = z_c$ and FB-regularity holds vacuously since there is no nonzero vector satisfying $z_c = 0, z_P > 0, z_N < 0$.

($\Rightarrow$) Conversely, let $x$ be FB-regular such that $\nabla \theta_{FB}(x) = 0$. The stationarity condition may then be written as

$$D_a F_{FB} + JF(x)^T D_b F_{FB} = 0$$

implying that

$$y^T D_a F_{FB} + y^T JF(x)^T D_b F_{FB} = 0 \quad (\ast)$$

for any $y \in \mathbb{R}^n$. 
Let us proceed by contradiction; suppose $x$ is not a solution to NCP($F$). Then, the set $\mathcal{R}$ is nonempty (some indices are non-complementary). Then $i \in \mathcal{R}$ implies that $z \equiv D_b F_{FB}$ is a nonzero vector by noting that

$$(D_a F_{FB})_i = 0 \iff (D_b F_{FB})_i = 0 \iff i \in C.$$ 

Moreover, by FB-regularity, we have that $z_C = 0, z_P > 0, z_N < 0$. Recall that the components of $v \equiv D_a F_{FB}$ and $z = D_b F_{FB}$ have the same signs componentwise, we have that

$$y^T(D_a F_{FB}) = y_C^T(D_a F_{FB})_C + y_P^T(D_a F_{FB})_P + y_N^T(D_a F_{FB})_N > 0,$$

since $y_R \neq 0$ and $y_C = 0, y_P \geq 0, y_N \leq 0$.

Furthermore, $y^T JF(x)^T(D_b F_{FB}) = y^T JF(x)^T z \geq 0$, where the last inequality follows from FB-regularity.

Together, these contradict (*) and therefore $\mathcal{R} = \emptyset$ and $x$ is a solution of NCP($F$).

\footnote{Note that $x \in SOL(F)$ if and only if $\mathcal{R} = \emptyset$.}
Comments

• If $JF(x) \succeq 0$ at $x$, then the earlier definition may be employed to show that $x$ is FB-regular (set $y = z$).

• In fact, $z^T JF(x)y \geq 0$ is equivalent to
  \[
  \sum_{i \in \mathcal{P}} z_i (JF(x)y)_i + \sum_{i \in \mathcal{N}} z_i (JF(x)y)_i. 
  \]
  By noting that $z_{\mathcal{P}} > 0$ and $z_{\mathcal{N}} < 0$, we have that this inequality holds if
  \[
  \nabla F_i(x)^T y \geq 0, \quad \forall i \in \mathcal{P} \\
  \nabla F_i(x)^T y \leq 0, \quad \forall i \in \mathcal{N}. 
  \]

• Therefore a sufficient condition for $x$ to be an FB-regular point is that there exist a $y \neq 0$ with $y_C = 0, y_{\mathcal{P}} \geq 0, y_{\mathcal{N}} \leq 0$ along with the above two inequalities.

• This in turn is equivalent to the existence of a nonzero $(u_{\mathcal{P}}, u_{\mathcal{N}})$ such that
  \[
  \begin{pmatrix}
  J_{\mathcal{P}} F_{\mathcal{P}} & -J_{\mathcal{P}} F_{\mathcal{P}} \\
  -J_{\mathcal{P}} F_{\mathcal{N}} & J_{\mathcal{N}} F_{\mathcal{N}}
  \end{pmatrix}
  \begin{pmatrix}
  u_{\mathcal{P}} \\
  u_{\mathcal{N}}
  \end{pmatrix} \succeq 0, u_{\mathcal{P}}, u_{\mathcal{N}} \geq 0. 
  \]
• Recall that for a matrix $M \in \mathbb{R}^{n \times n}$ for which there exists a $u$ satisfying
\[ Mu \geq 0 \]
\[ u \geq 0, \]
is called an $S_0$ matrix (containing the class of S-matrices or Stieltje matrices.)

• **Advantage:** Computing whether this sufficient condition does indeed hold is a finite procedure and requires solving an LP while verifying FB-regularity cannot in general be shown using a finite procedure.

• To relate the matrix
\[ M = \begin{pmatrix} J_P F_P & -J_P F_P \\ -J_P F_N & J_N F_N \end{pmatrix} \]
to the Jacobian $JF(x)$, we use a sign matrix $\Lambda(x)$, a diagonal matrix whose diagonal entries $\lambda_i$ are defined as
\[ \lambda_i = \begin{cases} 1, & i \in \mathcal{P} \\ -1, & i \in \mathcal{N} \\ 0, & i \in \mathcal{C}. \end{cases} \]
It follows that $M$ is a principal submatrix of $\Theta(x) = \Lambda(x) JF(x) \Lambda(x)$ with rows and columns corresponding to $\mathcal{R} = \mathcal{P} \cup \mathcal{N}$ of the vector $x$. 
This leads to the following definition:

**Definition 2** Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable. We say that $JF(x)$ is a signed $S_0$ matrix if $\Theta(x)_{RR}$ is an $S_0$ matrix. Furthermore, we say that $F$ has **differentiable signed $S_0$** property at $x$ if $JF(x)$ is a signed $S_0$ matrix. If $F$ has the differentiable signed $S_0$ property at every point in its domain, then $F$ is a signed $S_0$ function.

Therefore the signed $S_0$ property of $JF(x)$ is a sufficient condition for $x$ to be FB-regular. Furthermore, even singular stationary points of $\theta_{FB}$ can be shown to be solutions to the NCP as the following corollary specifies.

**Corollary 1** Suppose that $F$ is continuously differentiable. If $F$ has the differentiable signed $S_0$ property at every singular stationary point, then every stationary point of $\theta_{FB}$ is a solution to NCP($F$). In this case, every accumulation point of the sequence of iterates produced by the (FBLSA) solves the NCP.
Nonsingularity of Newton Approximation

- the linear Newton approximation, denoted by $T$, should be chosen such that
  \[ T(x) \subseteq \mathcal{D}_a(x) + \mathcal{D}_b(x)JF(x). \]

- What \textbf{sufficient} conditions do we need to ensure that all matrices in $\mathcal{D}_a(x) + \mathcal{D}_b(x)JF(x)$, we have nonsingularity.

- While nonsingularity doesn't guarantee that the sufficient descent criterion will be met, it does allow for \textbf{fast local convergence} (see Th. 7.5.15)

- To ensure that $\mathcal{D}_a(x) + \mathcal{D}_b(x)JF(x)$ contains only nonsingular matrices, additional properties must be imposed on $F$.

**Lemma 1** Let $M \in \mathbb{R}^{n \times n}$ be a given matrix. The following two statements are equivalent:

(a) $M$ is a $P_0$ matrix
(b) Every matrix of the form \( D_a + D_b M \) is nonsingular for all nonnegative(nonpositive) diagonal matrices \( D_a \) and \( D_b \) with \( (D_a)_{ii} > (\leq) 0 \) for all \( i = 1, \ldots, n \).

**Theorem 2** Suppose that \( F : \mathbb{R}^n \to \mathbb{R}^n \) is continuously differentiable in a neighborhood of \( x \in \mathbb{R}^n \). Let \( M \equiv JF(x) \); also let \( \bar{\alpha} \equiv \gamma \cup \beta \cup \delta \) be the complement if \( \alpha \) in \( \{1, \ldots, n\} \), where

\[
\begin{align*}
\alpha & \equiv \{ i : x_i = 0 < F_i(x) \} \\
\beta & \equiv \{ i : x_i = 0 = F_i(x) \} \\
\gamma & \equiv \{ i : x_i > 0 = F_i(x) \} \\
\delta & \equiv \{1, \ldots, n\} \setminus (\alpha \cup \beta \cup \gamma).
\end{align*}
\]

Assume that

(a) the submatrices \( M_{\tilde{\gamma}\tilde{\gamma}} \) are nonsingular for all \( \tilde{\gamma} \) satisfying \( \gamma < \tilde{\gamma} < \gamma \cup \beta \).

(b) the Schur complement of \( M_{\gamma\gamma} \) in \( M_{\bar{\alpha}\bar{\alpha}} \) is a \( P_0 \) matrix.
Constrained Methods

- Consider a constrained formulation of the NCP:

\[
\begin{aligned}
(CNCP) \min & \quad \theta_{FB}(x) \\
\text{subject to} & \quad x \in \mathbb{R}^n_+,
\end{aligned}
\]

- Suppose that \( F \) is defined over \( \mathbb{R}^n_+ \) in (CNCP). Specifically, we consider algorithms that maintain nonnegativity of the iterates (for instance, \( F \) may lose differentiability outside \( \mathbb{R}^n_+ \)).

- Note that \( x \) is a stationary point of (CNCP) if \( 0 \leq x \perp \nabla \theta_{FB}(x) \geq 0 \) or \( \min(x, \nabla \theta_{FB}(x)) = 0 \).

**Theorem 3** Suppose that \( F : \Omega \subset \mathbb{R}^n_+ \rightarrow \mathbb{R}^n \) be continuously differentiable on the open set \( \Omega \) and that \( x \) is a constrained stationary point of (CNCP). Then \( x \) is a stationary point of \( NCP(F) \) if and only if \( x \) is a \( \text{FB-regular} \) point of \( \theta_{FB} \)

**Proof:**

- One direction (forward) is the same as the earlier theorem proved.
• We prove the other direction. Suppose that \( x \) is an FB-regular constrained stationary point but not a solution of NCP(F). Let \( I_0 \) and \( I_+ \) represent the following:

\[
I_0 \equiv \{ i : x_i = 0 \}
\]
\[
I_+ \equiv \{ i : x_i > 0 \}.
\]

• The stationarity conditions may then be written as

\[
(\nabla \theta_{FB}(x))_i = 0, i \in I_+
\]
\[
(\nabla \theta_{FB}(x))_i \geq 0, i \in I_0.
\]

• Let \( z = D_b F_{FB}(x) \) where \( D_b \) is an arbitrary diagonal matrix in \( D_b(x) \). Since \( x \notin SOL(F) \), \( z \neq 0 \). Then let \( y \) be the vector in the definition of FB-regularity. Then

\[
i \in I_0 \implies x_i = 0 \implies i \in \mathcal{C} \cup \mathcal{N} \implies y_i \leq 0.
\]

(note that \( i \in \mathcal{N} \) otherwise \( x \in SOL(F) \))

• Therefore,

\[
y^T \nabla \theta_{FB}(x) = y_{I_0}^T \nabla I_0 \theta_{FB}(x) \leq 0. \quad (**) \]

But from the definition, we have \( y^T JF(x)^T z \geq 0 \). Moreover \( y^T D_a F_{FB}(x) > 0 \) as in the earlier proof. But these two inequalities contradict (**) . This concludes the proof.
Algorithms

• Next, we discuss some algorithms for solving such problems.

• Given an iterate $x^k$, we compute the next iterate by solving $LCP(q^k, JF(x^k))$, where $q^k \equiv F(x^k) - JF(x^k)x^k$.

• Globalization of this algorithm is achieved by using the FB merit function along which we conduct a linesearch using the steepest descent direction.

• Such a direction is computed by solving the problem

$$\begin{cases} 
\min & \nabla \theta_{FB}(x^k)^T d + \frac{1}{2}d^T d \\
\text{subject to} & x^k + d \in \mathbb{R}^n_+.
\end{cases}$$

• It may be shown that

$$\hat{d}^k = \max(-x^k, -\nabla \theta_{FB}(x^k)) = -\min(x^k, \theta_{FB}(x^k)).$$

Moreover $\hat{d}^k = 0 \iff \min(x^k, \nabla \theta_{FB}(x^k)) = 0$. Thus if $x^k$ is not a stationary point of (CNCP), then $\hat{d}^k$ is a nonzero descent direction of $\theta_{FB}$ at $x^k$,

$$\nabla \theta_{FB}(x^k)^T \hat{d}^k + \frac{1}{2}(\hat{d}^k)^T \hat{d}^k < 0 \text{ implying that } \|\hat{d}^k\| < \sqrt{-2\nabla \theta_{FB}(x^k)^T \hat{d}^k}.$$
• Note that this direction is a fall-back direction.

• Constrained FB Linesearch Algorithm

0. $x^0, \rho > 0, p > 1, \gamma \in (0, 1)$

1. $k = 0$

2. If $x^k$ is a stationary point of (CNCP), stop.

3. Find a solution $y^{k+1}$ of the LCP $(q^k, JF(x^k))$ and set $d^k \equiv y^{k+1} - x^k$. If the LCP is not solvable or if $\nabla \theta_{FB}(x^k)^T d^k \leq -\rho \|d\|^p$, is not satisfied, set $d^k \equiv \hat{d}^k$.

4. Find the smallest nonnegative integer $i_k$ such that with $i = i_k$,

$$ \theta_{FB}(x^k + 2^{-i}d^k) \leq \theta_{FB}(x^k) - \min\{-2^{-i}\gamma \nabla \theta_{FB}(x^k)^T d^k, (1 - \gamma)\theta_{FB}(x^k)\}.$$ 

5. Set $x^{k+1} \equiv x^k + \tau_k d^k$ and $k \equiv k + 1$; go to 2.
Extensions

- Discuss extensions to NCPs
- Finite lower (or upper) bounds
- Mixed-complementarity problems
- Box-constrained VIs
Finite lower(or upper) bounds

- Consider the VI(K,F) in which the set $K$ is defined as

$$K \equiv \{ x \in \mathbb{R}^n : x \geq a \},$$

where $a$ is a given vector.

- One option is to define $y = x - a$ and convert the problem into an NCP. Alternate approach requires observing that

$$(x \in SOL(K, F)) \iff [x_i = a_i \implies F_i(x) \geq 0] \text{ and } [x_i > a_i \implies F_i(x) = 0].$$

- An equation reformulation for this VI is given by

$$0 = \mathbf{F}^\ell_{FB}(x) \equiv \begin{pmatrix}
\theta_{FB}(x_1 - a_1, F_1(x)) \\
\vdots \\
\theta_{FB}(x_n - a_n, F_n(x))
\end{pmatrix},$$

which immediately suggests the merit function given by $\theta^\ell_{FB}(x) = \frac{1}{2} (\mathbf{F}^\ell_{FB}(x))^T \mathbf{F}^\ell_{FB}(x)$.

- Similarly for upper bounds.
Mixed-complementarity Problems

• Consider the MiCP(G,H) given by

\[ \begin{align*}
G(u, v) &= 0, \quad u \text{ free} \\
0 &\leq v \perp H(u, v) \geq 0,
\end{align*} \]

where \( G : \mathbb{R}^n \to \mathbb{R}^{n_1} \) and \( H : \mathbb{R}^n \to \mathbb{R}^{n_1} \) such that \( n_1 + n_2 = n \).

• The FB-based equation formulation is given by

\[ 0 = F_{\text{MCP}}^{\text{FB}}(u, v) \equiv \begin{pmatrix} G(u, v) \\ \theta_{\text{FB}}(v_1, H_1(u, v)) \\ \vdots \\ \theta_{\text{FB}}(v_n, H_n(u, v)) \end{pmatrix}. \]

• The associated merit function is given by \( \theta_{\text{MCP}}^{\text{FB}}(u, v) = \frac{1}{2} F_{\text{FB}}^{\text{MCP}}(u, v)^T F_{\text{FB}}^{\text{MCP}}(u, v) \).
Key properties

**Proposition 2** Let $G, H \in C^2$ be as defined earlier. Then the following two statements are valid:

1. $F^\text{MCP}_{FB}$ is semismooth and $\theta_{FB}$ is continuously differentiable. If $G$ and $H$ have Lipschitz continuous derivatives, then $F^\text{MCP}_{FB}$ is strongly semismooth.

2. If $g_{FB}(a, b)$ is a linear Newton approximation of $\psi_{FB}(a, b)$, then a linear Newton approximation scheme of $F^\text{MCP}_{FB}$ is given by

$$A(u, v) = \left\{ \begin{pmatrix} J_u G & J_v G \\ D_b J_u H(u, v) & D_a + D_b J_v H(u, v) \end{pmatrix} \right\}$$

where $D_a$ and $D_b$ are $n_2 \times n_2$ diagonal matrices whose $i$th entries are the first and second elements of an element $\xi_i \in g_{FB}(v_i, H_i(u, v))$. If $g_{FB}(a, b) \subseteq \partial \psi_{FB}(a, b)$, then

$$\nabla \theta^\text{MCP}_{FB}(u, v) = A^T F^\text{MCP}_{FB}(u, v), \forall A \in A(u, v).$$
Box-constrained VIs

- Consider the box-constrained VI($K$, $F$) with
  
  \[ K \equiv \{ x \in \mathbb{R}^n : a_i \leq x \leq b_i, i = 1, \ldots, n \}. \]

- Just as in the 1-sided case, we have
  
  \[
  x_i = a_i \implies F_i \geq 0 \]
  
  \[
  a_i < x_i < b_i \implies F_i = 0 \]
  
  \[
  x_i = b_i \implies F_i \leq 0. \]

- Can we define a generalization of a C-function that captures the above properties. Specifically, given a pair of scalars $-\infty \leq \tau < \tau' \leq \infty$, we call a function $\phi(\tau', \tau, \ldots) : \mathbb{R}^2 \rightarrow \mathbb{R}$ a B-function if

  \[
  \phi(\tau', \tau, r, s) = 0 \iff \tau \leq r \leq \tau'
  \]

  and $(r, s)$ satisfies:

  \[
  r = \tau \implies s \geq 0
  \]

  \[
  \tau' > r > \tau \implies s = 0
  \]

  \[
  r = \tau' \implies s \leq 0.
  \]
• Note that a B-function with $\tau = 0$, $\tau = \infty$ is a C-function.

• Conversely given an C-function that satisfies the sign reversal property given by
\[
\psi(a, b) \leq 0 \implies ab \leq 0
\]
and any pair of scalars $-\infty \leq \tau < \tau' \leq \infty$, the function $\phi(\tau, \tau'; r, s)$ is defined by
\[
\phi(\tau, \tau'; r, s) = \begin{cases} 
\psi(\tau' - r, -s) & -\infty = \tau < \tau' < \infty \\
\psi(r - \tau, \psi(\tau' - r, -s)) & -\infty < \tau < \tau' < \infty \\
\psi(r - \tau, s) & -\infty = \tau < \tau' = \infty \\
s & -\infty = \tau < \tau' = \infty.
\end{cases}
\]

**Proof:**

• When one bound is finite, define a new variable (as shown earlier)

• Need to verify when both bounds are finite.

• In this case, we have
\[
\phi(\tau, \tau'; r, s) \iff 0 \leq r - \tau \perp \psi(\tau' - r, -s) \geq 0.
\]

• If $\tau < r$, then $\psi(\tau' - r, -s) = 0$ which implies that
\[
0 \leq \tau' - r \perp s \leq 0.
\]
• If \( \tau' > r \), implies that \( s = 0 \). If \( \tau' = r \), then \( s \leq 0 \).

• If \( \tau = r \), then \( \psi(\tau' - r, -s) \geq 0 \) and \( \tau' - r > 0 \). Therefore by the sign-reversal property, \( s \geq 0 \) and \( \phi \) is a B-function.