Lecture 19

Global Methods

October 28, 2008

Outline

- Line Search Algorithm
  - Gauss-Newton
- Application to a Complementarity Problem
- Trust Region Methods

Line Search for B-Differentiable Problems

Step 0 Choose $x^0 \in X$ and $\gamma \in (0,1)$. Set $k = 0$.

Step 1 If x^k is B-stationary point, stop.

Step 2 Choose a symmetric positive definite matrix H^k, and determine a vector d^k solving the direction-search problem with $x = x^k$ and $H = H^k$. Set $i = 0$.

Step 2(i) If

$$\theta(x^k + d^k/2^i) \le \theta(x^k) + \frac{\gamma}{2^i}\,\theta'(x^k; d^k),$$

set $i_k = i$ and $\tau_k = 2^i$. Otherwise, set $i := i + 1$ and repeat Step 2(i).

Step 3 Set $x^{k+1} = x^k + \tau_k d^k$, $k := k + 1$ and go to Step 1.

Convergence Result

Proposition 8.3.7. Let $X \subseteq \mathbb{R}^n$ be closed convex set and θ be B-differentiable on X. Let $\{x^k\}$ be a sequence generated by the Line Search Algorithm. Let $\{x^k \mid k \in \mathcal{K}\}$ be a subsequence such that

(a) There exist positive scalars c_1 and c_2 satisfying for every $k \in \mathcal{K}$,

$$c_1\|y\|^2 \le y^T H^k y \le c_2\|y\|^2 \qquad \text{for all } y \in \mathbb{R}^n$$

(b) The subsequence $\{x^k \mid k \in \mathcal{K}\}$ converges to a vector x^*

(c) The function θ has a strong F-derivative at x^*, i.e.,

$$\lim_{\substack{y \ne z \\ (y,z)\to(x^*,x^*)}} \frac{\mathrm{e}(y) - \mathrm{e}(z)}{\|y - z\|} = 0,$$

where $\mathrm{e}(y) = \theta(y) - \theta(x^*) - \nabla\theta(x^*)^T(y - x^*)$

Then, x^* is a B-stationary point of the minimization problem $\min_{x\in X} \theta(x)$.

Case of Differentiable θ

Let $X \subseteq \mathbb{R}^n$ be closed and convex and θ be continuously differentiable. Assume that we choose matrices H^k so that for some scalars $c_1 > 0$ and $c_2 > 0$, and for all k, there holds:

$$c_1\|y\|^2 \le y^T H^k y \le c_2\|y\|^2 \qquad \text{for all } y \in \mathbb{R}^n.$$

Let $\{x^k\}$ be generated by Line Search Algorithm with such matrices H^k.

Then, by Proposition 8.3.7, **every accumulation point $\tilde{x}$ of $\{x^k\}$ is a stationary point** of the problem $\min_{x\in X} \theta(x)$:

$$\nabla\theta(\tilde{x})^T(x - \tilde{x}) \ge 0 \qquad \text{for all } x \in X.$$

Caution: Proposition says **nothing about the existence** of an accumulation point of $\{x^k\}$. It merely says what **property an accumulation point would have** if it existed.

Gauss-Newton Method

We apply the Line Search Algorithm to find an (unconstrained) zero of the following system

$$G(x) = 0 \qquad x \in \mathbb{R}^n$$

where $G : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable.

We define $\theta(x)$ by

$$\theta(x) = \frac{1}{2}\|G(x)\|^2 = \frac{1}{2}G(x)^T G(x)$$

We discuss how to **exploit the special structure** of θ for defining the matrix H in the **direction-search problem**

$$\begin{array}{ll} \text{minimize} & \theta'(x;d) + \frac{1}{2} d^T H d \\ \text{subject to} & d \in X - x \end{array}$$

The modification we are about to see is a modification of well-known **Gauss-Newton method** for solving the system $G(x) = 0$.

Modified Gauss-Newton Method

In the Line Search Algorithm for a given x^k, use the matrix H^k given by

$$H^k = JG(x^k)^T JG(x^k) + \|G(x^k)\| \; I,$$

where I is the $n \times n$-identity matrix.

Note that H^k is **positive definite when $G(x^k) \neq 0$**.

The direction-search problem corresponds to the unconstrained minimization of the (strictly) convex function

$$\nabla\theta(x^k)^T d + \frac{1}{2} dH^k d.$$

The solution d satisfies: $\quad \nabla\theta(x^k) + H^k d = 0 \quad \Leftrightarrow \quad H^k d = -\nabla\theta(x^k)$

Using the expression for H^k and $\nabla\theta(x) = JG(x)^T G(x)$, we obtain

$$\left(JG(x^k)^T JG(x^k) + \|G(x^k)\| \, I\right) \; d = -JG(x^k)^T G(x^k)$$

Hence: $\quad d = -\left(JG(x^k)^T JG(x^k) + \|G(x^k)\| \, I\right)^{-1} \; JG(x^k)^T G(x^k)$

Gauss-Newton: $d = -\left(JG(x^k)^T JG(x^k)\right)^{-1} JG(x^k)^T G(x^k)$

Modified Gauss-Newton: Convergence Result

Proposition 8.3.8: Let $G : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable. Then, every accumulation point of the modified Gauss-Newton method is stationary point of the unconstrained problem

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} G(x)^T G(x).$$

Proof: Suppose x^* is an accumulation point of $\{x^k\}$. Suppose that $G(x^*) = 0$. Then, x^* is a global minimum of $\theta(x)$ over $\mathbb{R}^n$, and hence it is a stationary point of $\theta(x)$. When $G(x^*) \neq 0$, we apply Proposition 8.3.7 - Homework 4. $\square$.

NOTE: Here, a stationary point $\tilde{x}$ of $\theta(x)$ satisfies

$$\nabla \theta(\tilde{x}) = 0 \qquad \Longleftrightarrow \qquad JG(\tilde{x})^T G(\tilde{x}) = 0$$

Again, the accumulation points are only characterized; they need not exist

Application to a Complementarity Problem

Applying the Line Search Method to constrained complementarity problem:
Given a closed convex set $X \subseteq \mathbb{R}^n$, **find an** $x \in X$ **such that**

$$0 \le F(x) \perp G(x) \ge 0,$$

where $F, G : \mathbb{R}^n \to \mathbb{R}^n$ are continuously differentiable.

- We define B-differentiable function θ as follows

$$\theta(x) = \frac{1}{2}\|\Phi(x)\|^2 \qquad \text{with} \quad \Phi(x) = \min\{F(x),\ G(x)\} \quad \text{for } x \in \mathbb{R}^n.$$

- Function Φ has components $\Phi_i : \mathbb{R}^n \to \mathbb{R}$ given by $\Phi_i = \min\{F_i(x),\ G_i(x)\}$. Thus, we have

$$\theta(x) = \frac{1}{2}\sum_{i=1}^{n}[\Phi_i(x)]^2$$

- Therefore,

$$\theta'(x;d) = \sum_{i=1}^{n} \Phi_i(x)\,\Phi_i'(x;d) \qquad \text{for all } x, d \in \mathbb{R}^n$$

where

$$\Phi_i'(x;d) = \begin{cases} F_i'(x;d) & \text{if } F_i(x) < G_i(x) \\ G_i'(x;d) & \text{if } G_i(x) < F_i(x) \\ \min\{F_i'(x;d), G_i'(x;d)\} & \text{if } F_i(x) = G_i(x) \end{cases}$$

By differentiability of F and G, we have $F_i'(x;d) = \nabla F_i(x)^T d$ and similarly $G_i'(x;d) = \nabla G_i(x)^T d$.

- Define the following **index** sets, for a fixed but arbitrary x,

$$\mathcal{I}_F(x) = \{i \mid F_i(x) < G_i(x)\}$$

$$\mathcal{I}_G(x) = \{i \mid G_i(x) < F_i(x)\}$$

$$\mathcal{I}_=(x) = \{i \mid F_i(x) = G_i(x)\}$$

Thus, we can write $\theta'(x;d)$ as follows

$$\theta'(x;d) = \sum_{i\in\mathcal{I}_F(x)} F_i(x)\nabla F_i(x)^T d + \sum_{i\in\mathcal{I}_G(x)} G_i(x)\nabla G_i(x)^T d + \sum_{i\in\mathcal{I}_=(x)} \min\left\{F_i(x)\nabla F_i(x)^T d, G_i(x)\nabla G_i(x)^T d\right\}$$

- The direction-search problem now involves a nonconvex function $\theta'(x;d)$ so the **convexity of this problem is lost**.

- To deal with the nonconvexity, we modify the direction-search problem by introducing a convex function $a(x, \cdot)$ defined as follows:

$$a(x,d) = \sum_{i \in \mathcal{I}_F(x)} F_i(x) \nabla F_i(x)^T d + \sum_{i \in \mathcal{I}_G(x)} G_i(x) \nabla G_i(x)^T d + \sum_{i \in \mathcal{I}_=(x)} \max\left\{F_i(x)\nabla F_i(x)^T d, G_i(x) \nabla G_i(x)^T d\right\}$$

- In fact, this function is **piecewise linear in d** for a fixed x.
- Furthermore, it is a **majorant** of the directional derivative $\theta'(x;\cdot)$ over $\mathbb{R}^n$, i.e., for every $x \in \mathbb{R}^n$,

$$a(x,d) \ge \theta'(x;d) \qquad \text{for all } d \in \mathbb{R}^n.$$

- Alternative convex majorant of θ can be obtained by replacing the last summand in the definition of the function $a(\cdot,\cdot)$ by the average

$$\frac{1}{2}\left(F_i(x)\nabla F_i(x)^T d + G_i(x)\nabla G_i(x)^T d\right)$$

Limsup Property of a

Proposition 8.3.20: Let $\{x^k, d^k\}$ be a convergent sequence. Then, for every sequence $\{t_k\}$ of positive scalars, we have

$$\limsup_{k\to\infty} \frac{\theta(x^k + t_k d^k) - \theta(x^k) - t_k\, a(x^k, d^k)}{t_k} \leq 0$$

Proof: We at first write the function $a(x,d)$ as the following sum:

$$a(x,d) = \sum_{i\in\mathcal{I}_F(x)} a_i(x,d) \; + \sum_{i\in\mathcal{I}_G(x)} a_i(x,d) \; + \sum_{i\in\mathcal{I}_=(x)} a_i(x,d)$$

where

$$a_i(x,d)) = \begin{cases} F_i(x)\nabla F_i(x)^T d & \text{if } i \in \mathcal{I}_F(x) \\ G_i(x)\nabla G_i(x)^T d & \text{if } i \in \mathcal{I}_G(x) \\ \max\{F_i(x)\nabla F_i(x)^T d, G_i(x)\nabla G_i(x)^T d\} & \text{if } i \in \mathcal{I}_=(x) \end{cases} \quad (1)$$

By definition, we have $\theta(x) = \sum_{i=1}^{n} [\Phi_i(x)]^2$. Fix an index i and for arbitrary k, consider the difference $[\Phi_i(x^k + t_k d^k)]^2 - [\Phi_i(x^k)]^2$.

Suppose $i \in \mathcal{I}_=(x^k)$, i.e., $F_i(x^k) = G_i(x^k)$ and without loss of generality assume that $\Phi_i(x^k) = F_i(x^k)$. Then, by relation (1) for a_i, we have

$$
\begin{aligned}
[\Phi_i(x^k + t_k d^k)]^2 - [\Phi_i(x^k)]^2 &= [F_i(x^k + t_k d^k)]^2 - [F_i(x^k)]^2 \\
&= 2t_k\, F_i(x^k) \nabla F_i(x^k)^T d^k + o(t_k) \\
&\le 2t_k\, a_i(x^k, d^k) + o(t_k)
\end{aligned}
$$

Suppose $i \in \mathcal{I}_F(x^k)$, so that $\Phi_i(x^k) = F_i(x^k)$. Then, similar to the preceding by relation (1) for a_i, we obtain

$$
\begin{aligned}
[\Phi_i(x^k + t_k d^k)]^2 - [\Phi_i(x^k)]^2 &= [F_i(x^k + t_k d^k)]^2 - [F_i(x^k)]^2 \\
&= 2t_k\, F_i(x^k) \nabla F_i(x^k)^T d^k + o(t_k) \\
&= 2t_k\, a_i(x^k, d^k) + o(t_k)
\end{aligned}
$$

Suppose $i \in \mathcal{I}_G(x^k)$. By the same line of argument with G_i in place of F_i, we conclude that

$$[\Phi_i(x^k + t_k d^k)]^2 - [\Phi_i(x^k)]^2 = 2t_k\, a_i(x^k, d^k) + o(t_k).$$

Therefore, for any i and k, there holds

$$[\Phi_i(x^k + t_k d^k)]^2 - [\Phi_i(x^k)]^2 \leq 2t_k\, a_i(x^k, d^k) + o(t_k),$$

implying that for all i and k,

$$\frac{1}{2}\left([\Phi_i(x^k + t_k d^k)]^2 - [\Phi_i(x^k)]^2 - t_k\, a_i(x^k, d^k)\right) \leq \frac{1}{2}\, o(t_k).$$

By summing over i, we obtain for all k,

$$\theta(x^k + t_k d^k) - \theta(x^k) - t_k\, a(x^k, d^k) \leq \frac{1}{2}\, o(t_k).$$

By dividing with t_k and taking $limsup$ as $k \to \infty$, we obtain the desired result. $\square$

Line Search Algorithm

Suppose we use Line Search Algorithm with

- Symmetric positive definite matrices H^k with uniform bounds on the minimum and the maximum eigenvalues, i.e., such that for some scalars $c_1 > 0$ and $c_2 > 0$, we have for all k
$$c_1\|y\|^2 \leq y^T H^k y \leq c_2\|y\|^2.$$
- Forcing function $\sigma(x, d) = -a(x, d)$
- At iteration k, the direction d^k determined by solving the following direction-search (convex) problem:
$$\begin{array}{ll} \text{minimize} & a(x^k, d) + \frac{1}{2} d^T H^k d \\ \text{subject to} & d \in X - x \end{array}$$
- Stopping rule $a(x^k, d^k) = 0$
- Otherwise, we must have $a(x^k, d^k) < 0$, and we determine the next iterate x^{k+1} by using Armijo line search rule

Some Properties

Suppose that $\{x^k \mid k \in \mathcal{K}\}$ converges to x^*, which belongs to X by closedness of X. In view of the direction search problem, it follows that

$$a(x^k, d^k) + \frac{1}{2}(d^k)^T H^k d^k \leq 0,$$

implying $(d^k)^T H^k d^k \leq -2\, a(x^k, d^k)$. By the choice of H^k, we have

$$c_1 \|d^k\|^2 \leq -2\, a(x^k, d^k)$$

Therefore

$$\|d^k\| \leq \sqrt{-2c_1^{-1}\, a(x^k, d^k)} \tag{2}$$

Hence, $\{d^k \mid k \in \mathcal{K}\}$ is bounded, and let $\mathcal{K}' \subseteq \mathcal{K}$ be such that $\{d^k \mid k \in \mathcal{K}'\}$ is convergent. Then, by Proposition 8.3.20, we obtain

$$\limsup_{\substack{k\to\infty \\ k\in\mathcal{K}'}} \frac{\theta(x^k + t_k d^k) - \theta(x^k) - t_k\, a(x^k, d^k)}{t_k} \leq 0$$

Hence, Theorem 8.3.3 applies and thus, $a(x^k, d^k) \to 0$ as $k \to \infty, k \in \mathcal{K}'$. By Eq. (2), we also have $d^k \to 0$ as $k \to \infty, k \in \mathcal{K}'$.

Forcing Function Convergence Properties

Recall:

Theorem 8.3.3. Let $X \subseteq \mathbb{R}^n$ be a convex set and let θ be locally Lipschitz function on X. Let $\{x^k\}$ be a sequence generated by the Line Search Algorithm. Assume that $\{x^k \mid k \in \mathcal{K}\}$ is a subsequence with the following properties:

(1) The function value sequence $\{\theta(x^k) \mid k \in \mathcal{K}\}$ is bounded below.

(2) For every sequence of positive scalars $\{t_k \mid k \in \mathcal{K}\}$ converging to zero, there holds

$$\limsup_{\substack{k\to\infty \\ k\in\mathcal{K}}} \frac{\theta(x^k + t_k d^k) - \theta(x^k) + t_k \sigma(x^k, d^k)}{t_k} \leq 0$$

Then, we have

$$\lim_{\substack{k\to\infty \\ k\in\mathcal{K}}} \sigma(x^k, d^k) = 0.$$

Convergence Result

Proposition 8.3.21: Let $X \subseteq \mathbb{R}^n$ be closed convex, and $F, G : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable. Let $\{x^k\}$ be generated by Line Search Algorithm, and assume that x^* is its accumulation point. Then, x^* is a solution to the complementarity problem

$$0 \leq F(x) \perp G(x) \geq 0 \qquad \text{with } x \in X$$

if and only if for every partitioning of the index set $\mathcal{I}_=(x^*)$ into three mutually disjoint sets A_F, $A_=$ and A_G, there exists a vector $d^* \in X - x^*$ satisfying

$$\begin{aligned} F_i(x^*) + \nabla F_i(x^*)^T d^* = 0 &\quad \text{for all } i \in \mathcal{I}_F(x^*) \cup A_F \\ G_i(x^*) + \nabla G_i(x^*)^T d^* = 0 &\quad \text{for all } i \in \mathcal{I}_G(x^*) \cup A_G \\ [\Phi_i(x^*)]^2 + \max\{F_i(x^*)\nabla F_i(x^*)^T d^*,\, G_i(x^*)\nabla G_i(x^*)^T d^*\} \leq 0 &\quad \text{for all } i \in A_= \end{aligned} \tag{3}$$

Proof

Only If Part: Suppose that the given partition property of $\mathcal{I}_{=}(x^*)$ holds. We prove that x^* solves the CP. Since x^* is an accumulation point of $\{x^k\}$, we can find an appropriate subsequence $\{x^k \mid k \in \mathcal{K}\}$ such that

$$\mathcal{I}_F(x^k) = \mathcal{C}_F, \quad \mathcal{I}_G(x^k) = \mathcal{C}_G, \quad \mathcal{I}_{=}(x^k) = \mathcal{C}_{=} \quad \text{for all } k \in \mathcal{K}. \tag{4}$$

Then, by continuity of $F, \nabla F, G$ and ∇G, we obtain

$$\mathcal{I}_F(x^*) \subseteq \mathcal{C}_F, \quad \mathcal{I}_G(x^*) \subseteq \mathcal{C}_G, \quad \mathcal{C}_{=} \subseteq \mathcal{I}_{=}(x^*). \tag{5}$$

Define a partition of the index set $\mathcal{I}_{=}(x^*)$, as follows:

$$A_F = \mathcal{C}_F \setminus \mathcal{I}_F(x^*), \qquad A_G = \mathcal{C}_G \setminus \mathcal{I}_G(x^*), \qquad A_{=} = \mathcal{C}_{=}. \tag{6}$$

Let $d^* \in X - x^*$ be a vector satisfying Eq. (3). For any $t \in [0, 1]$, the vector $d = t(d^* + x^* - x^k)$ is feasible in direction-search problem at iteration k, implying that

$$a(x^k, d) + \frac{1}{2} d^T H^k d \geq a(x^k, d^k) + \frac{1}{2} (d^k)^T H^k d^k$$

We have $a(x^k, d^k) \to 0$ and $d^k \to 0$ as $k \to \infty,\ k \in \mathcal{K}$. Furthermore, by definition of a, we obtain $a(x^k, d) = t\, a(x^k, d^* + x^* - x^k)$. Thus,

$$t \lim_{\substack{k\to\infty \\ k\in\mathcal{K}}} a(x^k, d^* + x^* - x^k) + t^2 \limsup_{\substack{k\to\infty \\ k\in\mathcal{K}}} (d^*)^T H^k d^* \geq 0$$

By the uniform max and min eigenvalue boundedness property for H^k, we have $\limsup_{\substack{k\to\infty \\ k\in\mathcal{K}}} (d^*)^T H^k d^* \leq t^2\, c_2 \|d^*\|^2$. Hence

$$t \lim_{\substack{k\to\infty \\ k\in\mathcal{K}}} a(x^k, d^* + x^* - x^k) + t^2\, c_2 \|d^*\|^2 \geq 0.$$

Dividing by t, and letting $t \downarrow 0$, we obtain $\lim_{\substack{k\to\infty \\ k\in\mathcal{K}}} a(x^k, d^* + x^* - x^k) \geq 0$. By the definition of a and equation (4), due to taking limits in $a(x^k, d^* + x^* - x^k)$, we have

$$\sum_{i\in\mathcal{C}_F} F_i(x^*)\nabla F_i(x^*)^T d^* + \sum_{i\in\mathcal{C}_G} G_i(x^*)\nabla G_i(x^*)^T d^*$$
$$+ \sum_{i\in\mathcal{C}_=} \max\left\{F_i(x^*)\nabla F_i(x^*)^T d^*, G_i(x^*)\nabla G_i(x^*)^T d^*\right\} \geq 0$$

From the definition of the partition in Eq. (6) and given relation (3), we see that

$$F_i(x^*)\nabla F_i(x^*)^T d^* = -[F_i(x^*)]^2 \qquad \text{for all } i \in \mathcal{C}_F$$

$$G_i(x^*)\nabla G_i(x^*)^T d^* = -[G_i(x^*)]^2 \qquad \text{for all } i \in \mathcal{C}_G$$

while the last term is less than $-[\Phi_i(x^*)]^2$ for $i \in \mathcal{C}_=$. Using the preceding relations, Eq. (5), and ..., we can see that

$$-2\theta(x^*) \geq 0 \qquad \textbf{Homework 4}$$

implying that $\theta(x^*) = 0$.

Trust Region Algorithms

- Similar to Line Search Algorithms
- Differ mainly in the direction-search problem
  - Do not require positive definiteness of H^k
  - Use another parameter Δ to bound the length of directions d, ensuring the existence of the solution d^k
- One interpretation of the direction-search problem is that it **approximates locally** the objective function $\theta(x)$

$$\theta(x) \approx \theta(x^k) + \theta'(x^k; x - x^k) + \frac{1}{2}(x - x^k)^T H^k (x - x^k)$$

or in terms of a majorant/approximation $a(x, \cdot)$ for the derivative $\theta'(x^k; \cdot)$

$$\theta(x) \approx \theta(x^k) + a(x^k, x - x^k) + \frac{1}{2}(x - x^k)^T H^k (x - x^k)$$

- In Trust Region Methods, the right-hand side is taken as a local model of $\theta(x)$ that we **trust** within a ball $B(x^k, \Delta)$ for some $\Delta > 0$

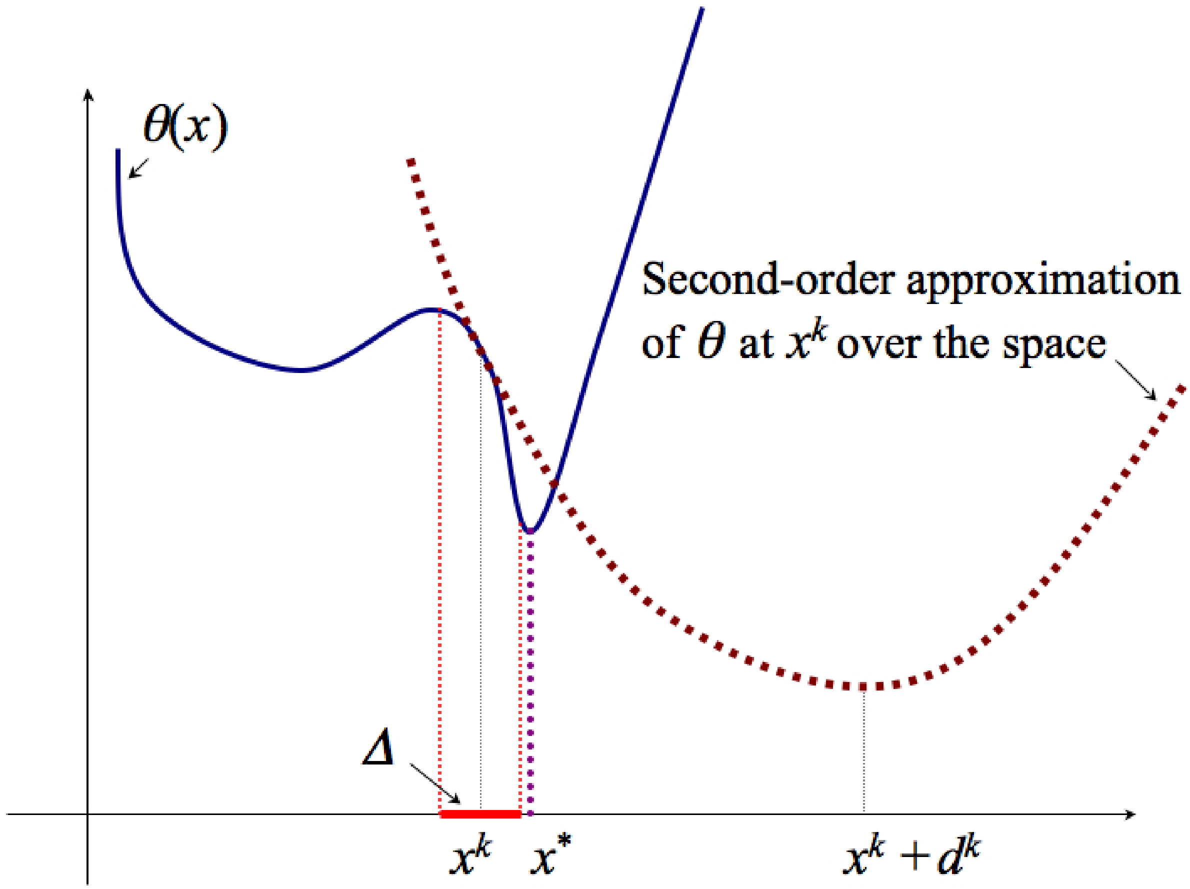
- We then minimize (approximately) the model function within the specified ball, and decide on x^{k+1} upon sufficient "descent"
  - We change the parameter Δ from iteration to iteration
  - Based on information gathered, we may decide to shrink or enlarge Δ
  - Depends on our belief of the quality of the local model within the Δ-ball

The idea of Trust Region

Let $\theta : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable, and suppose we want to minimize θ over $\mathbb{R}^n$. At a given iterate x^k, suppose we approximate the function with a second-order model

$$\theta(x^k + d) \approx \theta(x^k) + \nabla\theta(x^k)^T d + \frac{1}{2} d^T \nabla^2 \theta(x^k) d$$

- A natural idea is to minimize the right-hand side over $d \in \mathbb{R}^n$
- Potential sources of trouble
  - The quadratic function in d may be **unbounded below**, so minimizing the right-hand side is meaningless
  - Even if it is bounded below, and therefore, the minimizer d^k exists, the **quality** of d^k becomes an issue
  - Direction d^k may lie far out of the region where the quadratic model provides good local approximation for $\theta(x^k + d)$



A possible remedy is to bound the region over which we minimize,

$$\min_{\|d\| \leq \Delta} \left\{ \nabla \theta(x^k)^T d + \frac{1}{2} d^T \nabla^2 \theta(x^k) d \right\}$$

Generic Trust Region Problem

Let $X \subseteq \mathbb{R}^n$ be closed convex, and let $\theta : \mathbb{R}^n \to \mathbb{R}$. For a given $\bar{x} \in X$, a scalar $\Delta > 0$, and a **symmetric** matrix H, the direction $\bar{d}$ is computed by solving (possibly approximately) the following problem

$$\begin{array}{ll} \text{minimize} & a(\bar{x}, d) + \frac{1}{2} d^T H d \\ \text{subject to} & \|d\| \leq \Delta, \quad d \in X - \bar{x} \end{array}$$

- In theory, any $\|\cdot\|$ can be used to define the trust region
- In practice, mostly used are Euclidean norm and max-norm
- Trust Region problem plays a role similar to that of the direction-search problem in Line Search Methods
- Uses the forcing function $\sigma(x, d) = -[a(x, d) + d^T H d/2]$
- Has a major operational difference from the Line Search Methods
  - After a sufficient objective decrease, shrinks or enlarges the trust region parameter Δ and generates a new iterate x^{k+1}
  - After an insufficient decrease, the parameter Δ shrinks and the algorithm does not move, $x^{k+1} = x^k$

Generic Trust Region Algorithm

Step 0 Choose $x^0 \in X$ and the parameters $0 < \gamma_1 < \gamma_2 < 1$, $\Delta_0 > 0$, $\Delta_{\min} > 0$, $\rho \in (0, 1]$. Set $k = 0$.

Step 1 Find a vector d^k such that $\|d^k\| \le \Delta_k$, $d^k \in X - x^k$, and

$$a(x^k, d^k) + \frac{1}{2}(d^k)^T H d^k \le \rho \min_{\substack{\|d\|\le\Delta \\ d\in X-x^k}} \left\{a(x^k, d) + \frac{1}{2} d^T H d\right\}$$

Step 2 If $\sigma(x^k, d^k) = 0$ stop.

Step 3 In case of insufficient decrease, i.e., $\theta(x^k+d^k)-\theta(x^k) > -\gamma_1\,\sigma(x^k, d^k)$ set $x^{k+1} = x^k$, $\Delta_{k+1} = \Delta_k/2$, $k := k + 1$ and go to Step 1. Otherwise, set $x^{k+1} = x^k + d^k$, and

$$\Delta_{k+1} = \begin{cases} \max\{2\Delta_k, \Delta_{\min}\} & \text{if } \theta(x^k + d^k) - \theta(x^k) \le -\gamma_2\,\sigma(x^k, d^k) \\ \max\{\Delta_k, \Delta_{\min}\} & \text{otherwise} \end{cases}$$

set $k := k + 1$ and go to Step 1.