Introduction

• Course: Game theory: Models, Algorithms and Applications

• Focus on a variety of game-theoretic problem classes

• Restricted to problems with continuous strategy sets

• An optimization bias - cast game-theoretic problems as complementarity or variational inequality problems (or others)

• Useful viewpoint - provides us with a means to discuss existence and uniqueness of equilibria as well as tools to obtain such points
Example: Prisoner’s dilemma

- Two suspects are arrested and charged with a crime

- Explain consequences:
  - if neither confesses: both will get 1 month in jail
  - if both confess: both will get 6 months in jail
  - if one confesses, he will be released while the other will get 9 months

- players simultaneously choose actions

- receive payoffs based on decisions of all players

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<tr>
<th></th>
<th>Prisoner 2</th>
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<tbody>
<tr>
<td></td>
<td>Mum</td>
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<tr>
<td>Prisoner 1</td>
<td></td>
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<tr>
<td>Mum</td>
<td>(-1, -1)</td>
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<tr>
<td>Fink</td>
<td>(0, -9)</td>
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- Bimatrix game:
Normal form representation

- The players in the game, indexed by $i \in \{1, \ldots, N\}$
- Player $i$’s strategy $s_i \in S_i$, the strategy set of player $i$
- Payoff of player $i$ is given by $u_i(s_1, \ldots, s_N)$.

**Definition 1** The normal form representation of an $N$-player game specifying strategy spaces $S_1, \ldots, S_N$ and their payoffs $u_1, \ldots, u_N$. We denote this game by $G = \{S_1, \ldots, S_N; u_1, \ldots, u_n\}$.

- Game theory provides a unique prediction for each player’s strategy
- Predicted strategy should be *best response* to remaining players’ strategies
- Prediction sometimes called *strategically stable* or *self-enforcing* - no player wants to *unilaterally deviate* from his predicted strategy
- Such a set of strategies is called a *Nash equilibrium*
Definition 2 In the normal form game $G$, the strategies $s_1^*, \ldots, s_N^*$ are a Nash equilibrium if for each $i$,

$$u_i(s_1^*, \ldots, s_i^*, \ldots, s_N^*) \geq u_i(s_1^*, \ldots, s_i, \ldots, s_N^*)$$

for every $s_i \in S_i$; i.e., $s_i^* \in \max_{s_i \in S_i} u_i(s_1^*, \ldots, s_i, \ldots, s_N^*)$.

- If this does not hold for for some $i \in \bar{I}$, then a subset of players can profit from deviating

- In the Prisoner’s dilemma, both players playing **FINK** is a Nash equilibrium
Battle of the sexes

- Bimatrix game:

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<tr>
<td>Opera</td>
<td>Opera</td>
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<td>Baseball</td>
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- (Opera, Opera) and (Baseball, Baseball) are both Nash equilibria

- Lack of uniqueness
Motivating example: Duopolistic Cournot Model (1838)

• Predates Nash equilibrium ideas

• Firms 1 and 2 produce items $q_1$ and $q_2$; $Q = q_1 + q_2$

• Let $p(Q)$, the price, be defined as $p(Q) = a - Q$

• Cost of production is $C_i(q_i) = cq_i$ and $c < a$

Normal form:

• $S_i = [0, \infty), s_i \equiv q_i \geq 0$.

• $P(Q) = 0, Q > a \iff$ firms will not produce more than $a$

\[
q_1^* = \arg\max_{q_1} q_1(a - (q_1 + q_2^*)) - cq_1 \\
q_2^* = \arg\max_{q_2} q_2(a - (q_1^* + q_2)) - cq_2
\]
• Nash equilibrium in quantities:

\[ q_1^* = \frac{1}{2} (a - q_2^* - c) \]
\[ q_2^* = \frac{1}{2} (a - q_1^* - c) \]
\[ q_1^* = q_2^* = \frac{a - c}{3}. \]

• Intuition: If \( q_2^* = 0 \), \( q_1^* = \frac{a-c}{2} \), a monopolist level
Motivating example: Oligopolistic electricity markets

Framework

• Producers have access to a day-ahead market (ignore real-time market)

• Capacity-constrained Cournot bids in day-ahead market

• Variety of price functions for specifying prices
A 2-node Market

- Consider a 2-node electricity market. Suppose node 1 has a demand center while node 2 houses \( n \) electricity producers

- Producer \( i \) produces \( q_i \) MW during a specific hour with capacity \( C_i \)

- Producer \( i \)'s costs are \( c_i q_i + \frac{1}{2}d_i q_i^2 \).

- Transmission capacity of the link \((1, 2)\) is infinite

- Price of power at node 1 is given by \( p(Q) \), \( Q = \sum_{i=1}^{N} q_i \).
Nash Equilibrium in generation levels

- Producers can affect prices and their payoffs are dependent on the generation levels of the remaining firms

- Specifically, producer $i$ is faced by

\[
\begin{align*}
G_i(q^{-i}) & \quad \text{maximize} \quad p(Q)q_i - c_i q_i - \frac{1}{2}d_i q_i^2 \\
& \quad \text{subject to} \quad q_i \leq C_i \quad (\lambda_i) \\
& \qquad \quad q_i \geq 0,
\end{align*}
\]

where $q^{-i} = (q_j)_{j \neq i}$.

- Nash equilibrium in generation levels: $(q_1^*, \ldots, q_n^*)$ where $q_i^* \in \arg \max G_i(q^{*-i})$ for all $i$. 
Flow control games

- Multitude of applications on the internet
- A large number of users - inherently noncooperative in nature
- Demands for bandwidth lead to congestion
- Solution concept - a Nash equilibrium in flow rates, where each agent maximizes its utility less cost of sending flow
• User $i$ has utility given by $U_i(x_i) := \ln(1 + x_i) + d_i$

• Cost of sending flow is $\mathcal{P}(x_i, x^{-i}) := \frac{k_i x_i^2}{N - \sum_{i=1}^{m} x_i}$.

• Finally, flow decisions are required to stay feasible with respect to the capacity constraint $\sum_{i=1}^{N} x_i \leq M$. 
Nash Equilibrium in flow decisions

- Users maximize the benefit of sending flow by solving

\[
\begin{align*}
D_i(x^{-i}) &\quad \text{maximize } U_i(x_i) - P(x_i; x^{-i}) \\
&\quad \text{subject to } x_i \geq 0,
\end{align*}
\]

- Nash equilibrium in generation levels: \((x^*_1, \ldots, x^*_n)\) where \(x^*_i \in \arg \max G_i(x^{-i},*)\), for all \(i\) and \(\sum_{i=1}^{N} x_i \leq M\).
Mixed-Strategy Nash Equilibrium

- In a host of practical settings, strategy decisions are continuous variables (as opposed to discrete such as in the bimatrix games).
- Example: Mixed-strategy Nash equilibria

**Definition 3** In the normal form game $G = \{S_1, \ldots, S_N; u_1, \ldots, u_n\}$, let $S_i = \{s_{i1}, \ldots, s_{iK}\}$. Then a mixed-strategy for player $i$ is a distribution $\bar{p}_i = (p_{i1}, \ldots, p_{iK})$, where $0 \leq p_{ik} \leq 1$ for $k = 1, \ldots, K$ and $\sum_{k=1}^{K} p_{ik} = 1$.

- Intuitively - a mixed strategy specifies the randomization over the set of pure strategies
- A mixed-strategy Nash equilibrium is given by a set $(\bar{p}_1^*, \ldots, \bar{p}_K^*)$.
- Important - mixed-strategies result in continuous decision variables (the probability masses)
Our focus: Continuous Strategy Sets

- This course is devoted to settings where the strategies are continuous variables and payoff functions are twice continuously differentiable.*

- Why? - This allows for casting the resulting Nash equilibrium problem as a related variational inequality/complementarity problem (will be defined shortly).

*We may discuss applications where the differentiability assumption is relaxed.
A Complementarity reformulation

- Consider a Nash equilibrium problem in which player $i$ solves

$$
(Opt_i) \quad \min f_i(x_i; x^{-i}) \quad \forall i
$$

$$
\begin{align*}
x_i & \geq 0, \\
\end{align*}
$$

where $f_i$ is player $i$’s payoff function (convex and twice continuously differentiable).

- The sufficient optimality conditions of each player can be jointly stated as†

$$
\begin{align*}
0 \leq x_1 & \perp \nabla x_1 f_1(x_1; x^{-1}) \geq 0 \\
\vdots \\
0 \leq x_n & \perp \nabla x_N f_n(x_n; x^{-n}) \geq 0,
\end{align*}
$$

†The symbol $\perp$ implies complementarity; $x \perp y \implies [x]_i[y]_i = 0, \quad \forall i$
or compactly as $0 \leq x \perp F(x) \geq 0$, $F = \begin{pmatrix} \nabla x_1 f_1 \\ \vdots \\ \nabla x_N f_N \end{pmatrix}$.

- Such a problem is called a **nonlinear complementarity problem** and is defined as

**Definition 4** *Given a mapping $F : \mathbb{R}^n_+ \rightarrow \mathbb{R}^n$, the NCP($F$) is to find a vector $x \in \mathbb{R}^n_+$ satisfying $0 \leq x \perp F(x) \geq 0$.***
A Variational Inequality reformulation

• $x^*$ is a Nash equilibrium if and only if for each player $i$ we have

$$ (y_i - x_i)^T \nabla_{x_i} f_i(x_i; x^{-i}) \geq 0, \quad \forall y^i \geq 0. $$

• Concatenating these conditions we have

\[
\begin{align*}
V(I) & \quad (y_1 - x_1)^T \nabla_{x_1} f_1(x_1; x^{-1}) \geq 0, \quad \forall y^1 \geq 0 \\
& \quad \vdots \\
& \quad (y_N - x_N)^T \nabla_{x_N} f_N(x_N; x^{-N}) \geq 0, \quad \forall y^N \geq 0.
\end{align*}
\]

or equivalently $(y - x)^T F(x) \geq 0$, $F = \begin{pmatrix} \nabla_{x_1} f_1 \\ \vdots \\ \nabla_{x_N} f_N \end{pmatrix}$
Such a problem is called a **variational inequality problem** and is defined as follows:

**Definition 5** *Given a mapping $F : \mathbb{R}^n_+ \to \mathbb{R}^n$, the $VI(F, \mathbb{R}^n_+)$ is to find a vector $x \in \mathbb{R}^n_+$ satisfying $(y - x)^TF(x) \geq 0$ for all $y \in \mathbb{R}^n_+$.***
A Generalization of Nash

• An important generalization of the Nash solution concept pertains to the strategy sets being related. Specifically, player $i$ solves

$$(\text{GenOpt}_i) \quad \min f_i(x_i; x^{-i}) \quad x_i \in K(x^{-i}).$$

• The resulting variational inequality (called a quasi variational inequality (QVI) is given by

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<th>QVI</th>
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<tr>
<td>$(y_1 - x_1)^T \nabla_{x_1} f_1(x_1; x^{-1}) \geq 0, \quad \forall y^1 \in K(x^{-1})$</td>
</tr>
<tr>
<td>$\vdots$</td>
</tr>
<tr>
<td>$(y_N - x_N)^T \nabla_{x_N} f_N(x_N; x^{-N}) \geq 0, \quad \forall y^N \in K(x^{-N})$.</td>
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Stackelberg Equilibrium Problems

Stackelberg (1934) proposed a dynamic duopolistic model:

- Leader moves first
- Subordinate moves second

Examples:

- Automobile industry: GM is a leader while Ford and Chrysler may be considered to be followers
- Tolling in traffic networks: Network operator sets tolls while users respond and reach an equilibrium subject to tolls

Crucial idea: follower’s problem is a parametrized VI/CP
Stackelberg-Cournot

Setting:

- Firm 1 chooses quantity $q_1 \geq 0$

- Firm 2 observes $q_1$ and maximizes $q_2(a - (q_1 + q_2) - c)$.

- Firm 2’s best response is $q_2^* = \frac{a - q_1 - c}{2}$

- Firm 1 may then choose $q_1$ by solving $q_1(a - (q_1 + q_2) - c)$ where $q_2$ is given by $\frac{a - q_1 - c}{2}$

- This results in $q_1^* = \frac{a - c}{2}$ and $q_2^* = \frac{a - c}{4}$. 
More generally

• Suppose firm 1’s payoff function is $f(x, y)$

• Firm 2’s complementarity problem is $0 \leq y \perp F(y; x) \geq 0$.

• The resulting problem faced by firm 1 is a **mathematical program with complementarity/equilibrium constraints**

\[
\begin{align*}
\text{MPCC} & \quad \min_{x,y} \quad f(x, y) \\
& \text{subject to} \quad 0 \leq y \perp F(y; x) \geq 0.
\end{align*}
\]
Why is this difficult

- The constraints are nonsmooth (do not satisfy "regularity" conditions at any feasible point)

- Existing convergence theory for nonlinear programming does not hold
Focus of course

Three broad problem areas:

- Nash equilibrium problems \(\implies\) VIs/CPs
- Generalized Nash equilibrium problems \(\implies\) QVIs
- Stackelberg equilibrium problems \(\implies\) MPECs/MPCCs
Questions in each setting

• Existence/uniqueness of equilibria

• Construction of convergent algorithms for obtaining such equilibria (if they do indeed exist)
  • Global convergence
  • Local convergence
Books

The following books will be adhered to with volume 1 of the first set of references serving as the class text.

Lecture Outline - tentative

1. (B,BON) Background: Optimization theory and algorithms (convexity, first and second-order KKT conditions, regularity conditions, duality theory, quick survey of algorithms). (2)

2. Nash equilibrium models
   (a) (MWG, KON) Nash equilibrium: Background (1)
   (b) (FP-I,CPS) Applications and models: Linear complementarity problems (LCPs), nonlinear complementarity problems (NCPs) and variational inequalities (VIs) (2)
   (c) (FP-I,CPS) Existence and uniqueness of equilibria (3)
   (d) Algorithms:
      i. (CPS) Lemke’s method (pivot method for LCPs) (1)
      ii. (FP-II) Interior-point approaches (2)
      iii. (FP-II) Sequential linearization approaches (1)
3. Generalized Nash equilibrium models

(a) Applications and models: Quasi-variational inequalities (QVIs) and mixed complementarity problems (mCPs) (2)

(b) (CPS,FP-I) Existence and uniqueness of equilibria (2)

(c) Algorithms
   i. (FP-II) Penalization approaches (2)

4. Stackelberg equilibrium models

(a) (LPR) Applications and models: Mathematical programs with complementarity constraints (MPCCs) (2)

(b) (LPR) Strong stationarity and second-order conditions (2)

(c) Algorithms
   i. (LPR) Sequential quadratic programming approaches (2)
   ii. (LPR) Interior-smoothing and Interior-penalty approaches (2)
Grading - really painful

- HW 30%

- Examination 40%

- Project (present some papers) 30%