IE 598ns
Game theory: Models, Algorithms and Applications

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Introduction

- Course: Game theory: Models, Algorithms and Applications
- Focus on a variety of game-theoretic problem classes
- Restricted to problems with continuous strategy sets
- An optimization bias - cast game-theoretic problems as complementarity or variational inequality problems (or others)
- Useful viewpoint - provides us with a means to discuss existence and uniqueness of equilibria as well as tools to obtain such points

Example: Prisoner's dilemma

- Two suspects are arrested and charged with a crime
- Explain consequences:
  - if neither confesses: both will get 1 month in jail
  - if both confess: both will get 6 months in jail
  - if one confesses, he will be released while the other will get 9 months
- players simultaneously choose actions
- receive payoffs based on decisions of all players
- Bimatrix game:

		Prisoner 2	
		Mum	Fink
Prisoner 1	Mum	(-1,-1)	(-9,0)
	Fink	(0,-9)	(-6,-6)

Normal form representation

- The players in the game, indexed by $i \in \{1, \ldots, N\}$
- Player i's strategy $s_i \in S_i$, the strategy set of player i
- Payoff of player i is given by $u_i(s_1, \ldots, s_N)$.

Definition 1 *The normal form representation of an* $N-$*player game specifying strategy spaces* $S_1, \ldots, S_N$ *and their payoffs* $u_1, \ldots, u_N$*. We denote this game by* $G = \{S_1, \ldots, S_N; u_1, \ldots, u_n\}$*.*

- Game theory provides a unique prediction for each player's strategy
- Predicted strategy should be *best response* to remaining players' strategies
- Prediction sometimes called **strategically stable** or **self-enforcing** - no player wants to **unilaterally deviate** from his predicted strategy
- Such a set of strategies is called a **Nash equilibrium**

Nash equilibrium

Definition 2 *In the normal form game* G*, the strategies* $s_1^*, \ldots, s_N^*$ *are a Nash equilibrium if for each* i*,*

$$u_i(s_1^*, \ldots, s_i^*, \ldots, s_N^*) \geq u_i(s_1^*, \ldots, s_i, \ldots, s_N^*)$$

for every $s_i \in S_i$*; i.e.,* $s_i^* \in \max_{s_i \in S_i} u_i(s_1^*, \ldots, s_i, \ldots, s_N^*)$.

- If this does not hold for for some $i \in \bar{I}$, then a subset of players can profit from deviating

- In the Prisoner's dilemma, both players playing FINK is a Nash equilibrium

Battle of the sexes

- Bimatrix game:

		Pat	
		Opera	Baseball
Chris	Opera	(2,1)	(0,0)
	Baseball	(0,0)	(1,2)

- (Opera,Opera) and (Baseball,Baseball) are both Nash equilibria

- Lack of uniqueness

Motivating example: Duopolistic Cournot Model (1838)

- Predates Nash equilibrium ideas
- Firms 1 and 2 produce items q_1 and q_2; $Q = q_1 + q_2$
- Let $p(Q)$, the price, be defined as $p(Q) = a - Q$
- Cost of production is $C_i(q_i) = cq_i$ and $c < a$ Normal form:
  - $S_i = [0, \infty)$, $s_i \equiv q_i \geq 0$.
  - $P(Q) = 0, Q > a \implies$ firms will not produce more than a

$$q_1^* = \arg\max_{q_1} \quad q_1(a - (q_1 + q_2^*)) - cq_1$$
$$q_2^* = \arg\max_{q_2} \quad q_2(a - (q_1^* + q_2)) - cq_2$$

- Nash equilibrium in quantities:

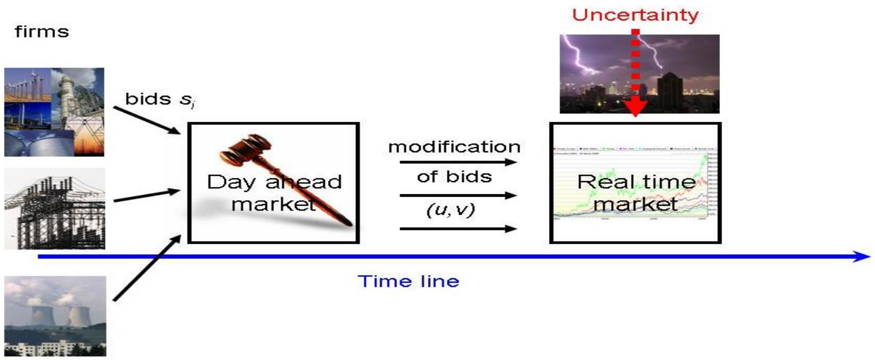
$$q_1^* = \frac{1}{2}(a - q_2^* - c)$$
$$q_2^* = \frac{1}{2}(a - q_1^* - c)$$
$$q_1^* = q_2^* = \frac{a-c}{3}.$$

- Intuition: If $q_2^* \equiv 0$, $q_1^* = \frac{a-c}{2}$, a monopolist level

Motivating example: Oligopolistic electricity markets

Framework

- Producers have access to a day-ahead market (ignore real-time market)
- Capacity-constrained Cournot bids in day-ahead market
- Variety of price functions for specifying prices



A 2-node Market

- Consider a 2-node electricity market. Suppose node 1 has a demand center while node 2 houses n electricity producers

- Producer i produces q_i MW during a specific hour with capacity C_i

- Producer i's costs are $c_i q_i + \frac{1}{2} d_i q_i^2$.

- Transmission capacity of the link $(1, 2)$ is infinite

- Price of power at node 1 is given by $p(Q)$, $Q = \sum_{i=1}^{N} q_i$.

Nash Equilibrium in generation levels

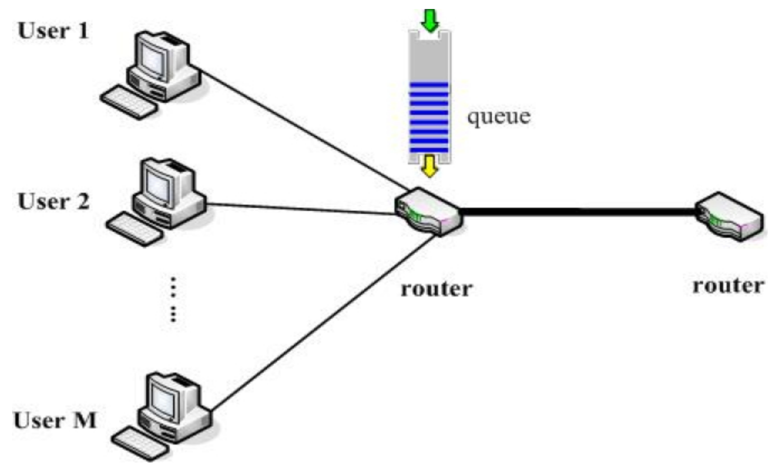
- Producers can affect prices and their payoffs are dependent on the generation levels of the remaining firms

- Specifically, producer i is faced by

$$\mathsf{G}_i(q^{-i}) \qquad \begin{array}{ll} \underset{q_i}{\text{maximize}} & p(Q)q_i - c_i q_i - \frac{1}{2} d_i q_i^2 \\ \text{subject to} & q_i \leq C_i \quad (\lambda_i) \\ & q_i \geq 0, \end{array}$$

where $q^{-i} = (q_j)_{j \neq i}$.

- Nash equilibrium in generation levels: $(q_1^*, \ldots, q_n^*)$ where $q_i^* \in \arg\max G_i(q^{*-i})$ for all i.



Flow control games

- Multitude of applications on the internet
- A large number of users - inherently noncooperative in nature
- Demands for bandwidth lead to congestion
- Solution concept - a Nash equilibrium in flow rates, where each agent maximizes its utility less cost of sending flow

- User i has utility given by $U_i(x_i) := \ln(1 + x_i) + d_i$
- Cost of sending flow is $\mathcal{P}(x_i, x^{-i}) := \frac{k_i x_i^2}{m - \sum_{i=1}^{N} x_i}$.
- Finally, flow decisions are required to stay feasible with respect to the capacity constraint $\sum_{i=1}^{N} x_i \leq M$.

Nash Equilibrium in flow decisions

- Users maximize the benefit of sending flow by solving

$$
\mathrm{D}_i(x^{-i}) \qquad \begin{array}{ll} \underset{x_i}{\text{maximize}} & U_i(x_i) - P(x_i; x^{-i}) \\ \text{subject to} & x_i \geq 0, \end{array}
$$

- Nash equilibrium in generation levels: $(x_1^*, \ldots, x_n^*)$ where $x_i^* \in \arg\max G_i(x^{-i,*})$, for all i and $\sum_{i=1}^{N} x_i \leq M$.

Mixed-Strategy Nash Equilibrium

- In a host of practical settings, strategy decisions are continous variables (as opposed to discrete such as in the bimatrix games).
- Example: Mixed-strategy Nash equilibria
- **Definition 3** *In the normal form game* $G = \{S_1, \ldots, S_N; u_1, \ldots, u_n\}$*, let* $S_i = \{s_{i1}, \ldots, s_{iK}\}$*. Then a mixed-strategy for player* i *is a distribution* $\bar{p}_i = (p_{i1}, \ldots, p_{iK})$*, where* $0 \leq p_{ik} \leq 1$ *for* $k = 1, \ldots, K$ *and* $\sum_{k=1}^{K} p_{ik} = 1$.
- Intuitively - a mixed strategy specifies the randomization over the set of pure strategies
- A mixed-strategy Nash equilibrium is given by a set $(\bar{p}_1^*, \ldots, \bar{p}_K^*)$.
- Important - mixed-strategies result in continuous decision variables (the probability masses)

Our focus: Continuous Strategy Sets

- This course is devoted to settings where the strategies are continuous variables and payoff functions are twice continuously differentiable.*

- Why? - This allows for casting the resulting Nash equilibrium problem as a related variational inequality/complementarity problem (will be defined shortly).

*We may discuss applications where the differentiability assumption is relaxed.

A Complementarity reformulation

- Consider a Nash equilibrium problem in which player i solves

$$(Opt_i) \qquad \min \quad f_i(x_i; x^{-i}) \\ x_i \geq 0,$$

where f_i is player i's payoff function (convex and twice continuously differentiable).

- The sufficient optimality conditions of each player can be jointly stated as†

$$\text{CP} \qquad \begin{array}{c} 0 \leq x_1 \perp \nabla_{x_1} f_1(x_1; x^{-1}) \geq 0 \\ \vdots \\ 0 \leq x_n \perp \nabla_{x_N} f_n(x_n; x^{-n}) \geq 0, \end{array}$$

†The symbol $\perp$ implies complementarity; $x \perp y \implies [x]_i[y]_i = 0, \quad \forall i$

or compactly as $0 \leq x \perp F(x) \geq 0, F = \begin{pmatrix} \nabla_{x_1} f_1 \\ \vdots \\ \nabla_{x_N} f_N \end{pmatrix}$

- Such a problem is called a **nonlinear complementarity problem** and is defined as

Definition 4 *Given a mapping* $F : \mathbb{R}^n_+ \rightarrow \mathbb{R}^n$, *the NCP(F) is to find a vector* $x \in \mathbb{R}^n_+$ *satisfying* $0 \leq x \perp F(x) \geq 0$.

A Variational Inequality reformulation

- x^* is a Nash equilibrium if and only if for each player i we have

$$(y_i - x_i)^T \nabla_{x_i} f_i(x_i; x^{-i}) \geq 0, \qquad \forall y^i \geq 0.$$

- Concatenating these conditions we have

$$\text{VI} \qquad \begin{array}{rl} (y_1 - x_1)^T \nabla_{x_1} f_1(x_1; x^{-1}) \geq 0, & \qquad \forall y^1 \geq 0 \\ \vdots & \\ (y_N - x_N)^T \nabla_{x_N} f_N(x_N; x^{-N}) \geq 0, & \qquad \forall y^N \geq 0. \end{array}$$

or equivalently $(y - x)^T F(x) \geq 0, F = \begin{pmatrix} \nabla_{x_1} f_1 \\ \vdots \\ \nabla_{x_N} f_N \end{pmatrix}$

- Such a problem is called a **variational inequality problem** and is defined as follows:

Definition 5 *Given a mapping* $F : \mathbb{R}^n_+ \rightarrow \mathbb{R}^n$, *the* $VI(F, \mathbb{R}^n_+)$ *is to find a vector* $x \in \mathbb{R}^n_+$ *satisfying* $(y - x)^T F(x) \geq 0$ *for all* $y \in \mathbb{R}^n_+$.

A Generalization of Nash

- An important generalization of the Nash solution concept pertains to the strategy sets being related. Specifically, player i solves

$$(GenOpt_i) \qquad \min \quad f_i(x_i; x^{-i}) \\ x_i \in K(x^{-i}).$$

- The resulting variational inequality (called a **quasi variational inequality (QVI)** is given by

$$\text{QVI} \qquad \begin{array}{lr} (y_1 - x_1)^T \nabla_{x_1} f_1(x_1; x^{-1}) \geq 0, & \forall y^1 \in K(x^{-1}) \\ \vdots & \\ (y_N - x_N)^T \nabla_{x_N} f_N(x_N; x^{-N}) \geq 0, & \forall y^N \in K(x^{-N}). \end{array}$$

Stackelberg Equilibrium Problems

Stackelberg (1934) proposed a dynamic duopolistic model:

- Leader moves first
- Subordinate moves second

Examples:

- Automobile industry: GM is a leader while Ford and Chrysler may be considered to be followers
- Tolling in traffic networks: Network operator sets tolls while users respond and reach an equilibrium subject to tolls

Crucial idea: follower's problem is a **parametrized** VI/CP

Stackelberg-Cournot

Setting:

- Firm 1 chooses quantity $q_1 \geq 0$
- Firm 2 observes q_1 and maximizes $q_2(a - (q_1 + q_2) - c)$.
- Firm 2's best response is $q_2^* = \frac{a - q_1 - c}{2}$
- Firm 1 may then choose q_1 by solving $q_1(a - (q_1 + q_2) - c)$ where q_2 is given by $\frac{a - q_1 - c}{2}$
- This results in $q_1^* = \frac{a-c}{2}$ and $q_2^* = \frac{a-c}{4}$.

More generally

- Suppose firm 1's payoff function is $f(x, y)$

- Firm 2's complementarity problem is $0 \leq y \perp F(y; x) \geq 0$.

- The resulting problem faced by firm 1 is a **mathematical program with complementarity/equilibrium constraints**

MPCC
$$\begin{array}{ll} \underset{x,y}{\text{minimize}} & f(x, y) \\ \text{subject to} & 0 \leq y \perp F(y; x) \geq 0. \end{array}$$

Why is this difficult

- The constraints are nonsmooth (do not satisfy "regularity" conditions at any feasible point)

- Existing convergence theory for nonlinear programming does not hold

Focus of course

Three broad problem areas:

- Nash equilibrium problems $\Longrightarrow$ VIs/CPs

- Generalized Nash equilibrium problems $\Longrightarrow$ QVIs

- Stackelberg equilibrium problems $\Longrightarrow$ MPECs/MPCCs

Questions in each setting

- Existence/uniqueness of equilibria

- Construction of convergent algorithms for obtaining such equilibria (if they do indeed exist)
  - Global convergence
  - Local convergence

Books

The following books will be adhered to with volume 1 of the first set of references serving as the class text.

- (FP) Facchinei, F. and Pang, J-S, *Finite-dimensional variational inequalities and complementarity problems*, Vols. I and II. Springer Series in Operations Research. Springer-Verlag, New York, 2003
- (CPS) Cottle, R. W. and Pang, J-S and Stone. R. E., *The Linear Complementarity Problem*, Computer Science and Scientific Computing. Academic Press, Inc., Boston, MA, 1992.
- (LPR) Luo, Z-Q, Pang, J-S and Ralph, D., *Mathematical programs with Equilibrium Constraints*, Cambridge University Press, Cambridge, 1996.
- (MWG) Mas-Colell, A., Whinston, M.D. and Green, J.R. *Microeconomic Theory* , Oxford University Press, USA, 1995.
- (B) Bertsekas D. P., *Nonlinear Programming*, Athena Scientific, 1995.
- (BON) Bertsekas Dimitri P., Ozdaglar, A. and Nedich, A., *Convex Optimization and Analysis*, Athena Scientific, 2003.
- (KON) Konnov, I.V. *Equilibrium Models and Variational Inequalities*, Elsevier, 2007.

Lecture Outline - tentative

1. (B,BON) Background: Optimization theory and algorithms (convexity, first and second-order KKT conditions, regularity conditions, duality theory, quick survey of algorithms). (2)

2. Nash equilibrium models

   (a) (MWG, KON) Nash equilibrium: Background (1)

   (b) (FP-I,CPS) Applications and models: Linear complementarity problems (LCPs), nonlinear complementarity problems (NCPs) and variational inequalities (VIs) (2)

   (c) (FP-I,CPS) Existence and uniqueness of equilibria (3)

   (d) Algorithms:

      i. (CPS) Lemke's method (pivot method for LCPs) (1)

      ii. (FP-II) Interior-point approaches (2)

      iii. (FP-II) Sequential linearization approaches (1)

3. Generalized Nash equilibrium models

(a) Applications and models: Quasi-variational inequalities (QVIs) and mixed complementarity problems (mCPs) (2)

(b) (CPS,FP-I) Existence and uniqueness of equilibria (2)

(c) Algorithms

i. (FP-II) Penalization approaches (2)

4. Stackelberg equilibrium models

(a) (LPR) Applications and models: Mathematical programs with complementarity constraints (MPCCs) (2)

(b) (LPR) Strong stationarity and second-order conditions (2)

(c) Algorithms

i. (LPR) Sequential quadratic programming approaches (2)

ii. (LPR) Interior-smoothing and Interior-penalty approaches (2)

Grading - really painful

- HW 30%
- Examination 40%
- Project (present some papers) 30%