IE (598ns)
November 11, 2007
Nedić & Shanbhag

Homework 3
Due November 7, 2007

Exercise 2.9.22
Exercise 2.9.27
Exercise 2.9.31 parts (a) and (b) only
Exercise 7.6.11

Exercise 1. Recall that for $p \geq 1$, the l_p norm of a vector $x \in \mathbb{R}^n$ is given by

$$\|x\|_p = \left(\sum_{i=1}^{n} x_i^p \right)^p.$$

Consider the function $f(x) = \|x\|_p$, and show that its subdifferential $\partial f(x)$ at $x = 0$ is given by

$$\partial f(0) = \{x \in \mathbb{R}^n \mid \|x\|_q \leq 1\},$$

where q satisfies $1/p + 1/q = 1$.

Exercise 2. Let K be a closed convex set in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable function with uniformly bounded maximum eigenvalues of its Hessian, i.e.,

$$\sup_{x \in \mathbb{R}^n} l_{\max}(\nabla^2 f(x)) < \infty.$$

Consider the following nonlinear problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in K. \end{array}$$

(a) Let $\tau > 0$ be such that $I - \tau \nabla^2 f(x)$ is positive semidefinite for all $x \in \mathbb{R}^n$. Show that a vector x^* is a stationary point of the above problem if and only if the vector x^* is an unconstrained stationary point of the following function:

$$\psi(x) = \tau f(x) - \frac{1}{2} \tau^2 \; \nabla f(x)^T \nabla f(x) + \frac{1}{2} \, \text{dist}(x - \tau f(x), K)^2,$$

where $\text{dist}(y, K)$ is the distance from y to the set K, i.e., $\text{dist}(y, K) = \min_{z \in K} \|y - z\|$. Show further that

$$\nabla \psi(x) = \left(I - \tau \nabla^2 f(x)\right) \left(x - \Pi_K(x - \tau \nabla f(x))\right) \qquad \text{for all } x.$$

(b) Let $f(x) = q^T x + \frac{1}{2} x^T M x$, where $q \in \mathbb{R}^n$ and M is a symmetric positive semidefinite matrix. Suppose τ is a scalar such that $0 < \tau < 1/l_{\max}(M)$. Show that $M - \tau M^2$ is positive semidefinite and $\psi(x)$ is convex.

The preceding is a trimmed version of Exercise 7.6.9.

Exercise 3 (Computational behavior of Newton type methods)**.** Consider the solution of the complementarity problem given by

$$0 \leq x \perp Mx + q \geq 0,$$

where $M \succ 0$ and q is an arbitary vector. Such a problem may arise in Nash-Cournot (uncapacitated) problem we discussed earlier. Implement the piecewise smooth Newton method for solving such a class of problems - think about you could construct random positive definite matrices of a fixed size. Recall that to observe local behavior, look at the error estimates.

1. You do NOT need to report code but please provide clear iteration logs for at least one problem as well as summaries of the problems solved.
2. Provide some insights of the behavior if the matrix properties change (weaken monotonicity assumption, for instance).
3. Comment and possibly demonstrate how conventional Newton methods may fail on such problems (or display poor behavior)