IE (598ns)
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Homework 1
Due September 12, 2007

Exercise 1. Let $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, \quad x_2 \geq 0\}$.

(a) Is X a cone? Answer yes or no, and justify your answer.

(b) Find the dual cone of X?

(c) Find the normal cone $N(\hat{x}; X)$ and the tangent cone $T(\hat{x}; X)$ to the set X at each of the following choices for $\hat{x}$:

(i) $\hat{x} = (0, 0)$,
(ii) $\hat{x} = (0, 1)$.

Exercise 2. Let $X = (0, 0) \cup \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0, \quad x_2 > 0\}$.

(a) Is X a cone? Answer yes or no, and justify your answer.

(b) Find the dual cone of X?

(c) Find the normal cone $N(\hat{x}; X)$ and the tangent cone $T(\hat{x}; X)$ to the set X at $\hat{x} = (0, 0)$.

Exercise 3. Let X_1 and X_2 be two sets with and let $\hat{x} \in X_1 \cap X_2$. Prove that the following relations hold for the tangent cones and the normal cones of X_1, X_2 and $X_1 \cap X_2$ at the point $\hat{x}$:

$$T(\hat{x}; X_1 \cap X_2) \subseteq T(\hat{x}; X_1) \cap T(\hat{x} : X_2),$$
$$N(\hat{x}; X_1) \cap N(\hat{x}; X_2) \subseteq N(\hat{x}; X_1 \cap X_2).$$

Exercise 4. In this exercise, we use the notion of a (lower) level set of a function, defined as follows.

Definition 1. The *(lower) level set of a function* $f : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\{x \in \mathbb{R}^n \mid f(x) \leq \gamma\}.$$

When the function is convex, we refer to this set simply by a *level set*.

Consider now a convex function $f : \mathbb{R}^n \to \mathbb{R}$. Let $\hat{x}$ be a given vector in $\mathbb{R}^n$, and consider the level set X associated with $\hat{x}$, i.e.,

$$X = \{x \in \mathbb{R}^n \mid f(\hat{x}) \leq f(\hat{x})\}.$$

Prove the following:

(a) Show that the set X is convex.

(b) Assuming that f is continuously differentiable over $\mathbb{R}^n$, show that the tangent cone $T(\hat{x}; X)$ and the normal cone $N(\hat{x}; X)$ of the level set X at the point $\hat{x}$ are given by

$$T(\hat{x}; X) = \{d \mid \nabla f(\hat{x})' d \leq 0\},$$
$$N(\hat{x}; X) = \{\lambda \nabla f(\hat{x}) \mid \lambda \geq 0\}.$$

Hint: You may find it useful to exploit the convexity of f and the first-order Taylor expansion of f at $\hat{x}$.

Exercise 5. Here, we consider a special constraint qualification, defined as follows.

Definition 2. The *Mangasarian-Fromovitz constraint qualification* is said to hold at a point $\bar{x}$ for the constraints

$$\begin{aligned} h(x) &= 0 \\ g(x) &\geq 0 \end{aligned}$$

where $h : \mathbb{R}^n \to \mathbb{R}^r$ and $g : \mathbb{R}^n \to \mathbb{R}^l$ if and only if the gradients $\nabla h_j(\bar{x}), j = 1, \ldots, r$, are linearly independent and there exists a $p \in \mathbb{R}^n$ such that

$$\begin{aligned} \nabla g_k(\bar{x})^T p &> 0, \quad k \in \mathcal{I}(\bar{x}) \\ \nabla h_j(\bar{x})^T p &= 0, \quad j = 1, \ldots r. \end{aligned}$$

Prove that an optimization problem containing the constraints

$$\begin{aligned} F(x, y) &\geq 0 \\ y &\geq 0 \\ y^T F(x, y) &= 0 \end{aligned}$$

does not satisfy the MFCQ at any feasible point. Define the linear independence constraint qualification and comment on whether it can hold, given the earlier result.

Exercise 6. Consider the halfspace defined by $H = \{x \in \mathbb{R}^n : a'x + \alpha \geq 0\}$, where $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Formulate and solve the optimization problem of finding the point in H that has the smallest Euclidean norm. [N & W, 12.15]

Exercise 7. A fixed point of a mapping $h : \mathbb{R}^n \to \mathbb{R}^n$ is a vector z such that

$$h(z) = z.$$

Suppose LCP(q, M) defines the problem of finding a z such that

$$\begin{aligned} Mz + q &\geq 0 \\ z &\geq 0 \\ z^T(Mz + q) &= 0. \end{aligned}$$

Then show that if z solves LCP(q, M) if and only if z solves the fixed-point problem with mapping

$$h(z) := \max(0, -q + (I - M)z).$$