Homework 8: Cone Properties and Conical Problems

Exercise 1 This exercise establishes linear convergence rate for subgradient method. Consider the projected subgradient method with Polyak stepsize:

\[ x_{k+1} = P_X[x_k - \alpha_k s_k] \quad k = 0, 1, \ldots, \]

where \( X \) is a closed convex set, \( x_0 \in X \) is an initial iterate, \( s_k \in \partial f(x_k) \), and \( \alpha_k = \frac{f(x_k) - f^*}{\|s_k\|^2} \).

Assume that \( f \) has a sharp set of minima, i.e., the optimal set \( X^* \) is nonempty and there exists \( \eta > 0 \) such that

\[ f(x) - f^* \geq \eta \text{dist}(x, X^*) \quad \text{for all} \quad x \in X, \]

where \( \text{dist}(y, Y) \) denotes the distance from the vector \( y \) to the set \( Y \), i.e., \( \text{dist}(y, Y) = \inf_{z \in Y} \|y - z\| \).

(a) Show that the iterates \( \{x_k\} \) generated by the method satisfy for any \( x^* \in X^* \) and any \( k \geq 0 \),

\[ \|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - \frac{(f(x_k) - f^*)^2}{\|s_k\|^2}. \]

(b) Show that the subgradients \( \{s_k\} \) of the iterates are bounded by some scalar \( C \), i.e., for some \( C > 0 \) we have \( \|s_k\| \leq C \) for all \( k \).

(c) Using (a) and (b), and the sharp minima property of \( f \), show that

\[ \text{dist}(x_k, X^*) \leq q^k \text{dist}(x_0, X^*) \quad \text{for all} \quad k, \]

where \( q = \sqrt{1 - \eta^2 C^2} \).

Exercise 2 Let \( C \subseteq \mathbb{R}^n \) be a set with nonempty interior. Consider its dual cone \( C^* \).

(a) Show that for any \( \hat{x} \in \text{int} C \) and any \( \lambda \in C^* \), we have

\[ \lambda^T \hat{x} > 0. \]

(b) Show that the cone \( C^* \) is pointed.

(c) If \( C \) is a cone, and if for some nonzero \( \lambda \in \mathbb{R}^n \), we have

\[ \inf_{x \in \text{int} C} \lambda^T x \geq c \quad \text{for some scalar} \quad c, \]

then

\[ \inf_{x \in C} \lambda^T x = 0. \]

[Note that the preceding relation implies that \( \lambda \in C^* \).]

Exercise 3 Let \( K \) be a (nonempty) closed convex cone, and let \( K^* \) be its dual cone. Show that if \( K \) is pointed, then \( K^* \) has a nonempty interior.

[You may consider using the fact \( (K^*)^* = K \) when \( K \) is nonempty closed convex cone.]