Homework 8: Cone Properties and Conical Problems

Exercise 1 *This exercise establishes linear convergence rate for subgradient method. Consider the projected subgradient method with Polyak stepsize:*

$$x_{k+1} = P_X[x_k - \alpha_k\, s_k] \qquad k = 0, 1, \ldots,$$

where X is a closed convex set, $x_0 \in X$ is an initial iterate, $s_k \in \partial f(x_k)$, and $\alpha_k = \frac{f(x_k)-f^*}{\|s_k\|^2}$.

Assume that f has a sharp set of minima, i.e., the optimal set X^* is nonempty and there exists $\eta > 0$ such that

$$f(x) - f^* \geq \eta\, dist(x, X^*) \qquad \text{for all } x \in X,$$

where $dist(y, Y)$ denotes the distance from the vector y to the set Y, i.e., $dist(y, Y) = \inf_{z\in Y} \|y - z\|$.

(a) *Show that the iterates $\{x_k\}$ generated by the method satisfy for any $x^* \in X^*$ and any $k \geq 0$,*

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - \frac{(f(x_k) - f^*)^2}{\|s_k\|^2}.$$

(b) *Show that the subgradients $\{s_k\}$ of the iterates are bounded by some scalar C, i.e., for some $C > 0$ we have $\|s_k\| \leq C$ for all k.*

(c) *Using (a) and (b), and the sharp minima property of f, show that*

$$dist(x_k, X^*) \leq q^k\, dist(x_0, X^*) \qquad \text{for all } k,$$

where $q = \sqrt{1 - \frac{\eta^2}{C^2}}$.

Exercise 2 *Let $C \subseteq \mathcal{R}^n$ be a set with nonempty interior. Consider its dual cone C^*.*

(a) *Show that for any $\hat{x} \in \text{int}C$ and any $\lambda \in C^*$, we have*

$$\lambda^T \hat{x} > 0.$$

(b) *Show that the cone C^* is pointed.*

(c) *If C is a cone, and if for some nonzero $\lambda \in \mathcal{R}^n$, we have*

$$\inf_{x\in \text{int}C} \lambda^T x \geq c \qquad \text{for some scalar } c,$$

then

$$\inf_{x\in C} \lambda^T x = 0.$$

[Note that the preceding relation implies that $\lambda \in C^*$.]

Exercise 3 *Let K be a (nonempty) closed convex cone, and let K^* be its dual cone. Show that if K is pointed, then K^* has a nonempty interior.*
[You may consider using the fact $(K^*)^* = K$ when K is nonempty closed convex cone.]