Homework 7: Numerical Experiments, Subgradients

In the Numerical Experiments, x^0 denotes an initial iterate and x^k denotes the k-th iterate generated by a method.

Exercise 1 Gradient Descent Method

Consider the problem of minimizing the quadratic objective

$$f(x_1, x_2) = \frac{1}{2}\left(x_1^2 + 100x_2^2\right).$$

Clearly, the optimal point is $x^* = (0,0)$ *and the optimal value is* $f^* = 0$.

(a) *Consider the gradient method with initial point* $x^0 = (100, 1)$. *Simulate 50 iterations of the gradient method for each of the exact line search stepsize, given by*

$$\alpha_k = \operatorname*{argmin}_{\alpha \geq 0} f(x^k - \alpha \nabla f(x^k)).$$

(b) *Consider the decrements* $f(x^k) - f^*$ *as a measure of performance of the algorithms. Analyze the results as follows:*

  (i) *Generate a composite plot of the results obtained in part (a) by plotting the function decrements* $f(x^k) - f^*$ *versus the iteration index* k.

  (ii) *Interpret the results.*

Exercise 2 Newton's Method

Consider the problem of minimizing the following objective

$$f(x_1, x_2) = e^{x_1 + 30x_2 - 0.1} + e^{x_1 - 30x_2 - 0.1} + e^{-x_1 - 0.1}.$$

[This objective is a modification of the example on page 470 of Boyd and Vandenberghe's book.]

(a) *Use the first order optimality conditions to determine the optimal point* x^* *and the optimal function value* f^*.

(b) *Consider Newton's method with initial point* $x^0 = (30, 1)$. *Simulate 50 iterations of the method for the backtracking line search with the following parameters:*

  (1) $\sigma = 0.3$ *and* $\beta = 0.9$

  (2) $\sigma = 0.3$ *and* $\beta = 0.1$

(c) *Consider the decrements* $f(x^k) - f^*$ *as a measure of performance of the algorithms. Analyze the results as follows:*

  (i) *Generate a composite plot of the results obtained in parts (1)-(2) by plotting the function decrements* $f(x^k) - f^*$ *versus the iteration index* k.

  (ii) *Interpret the results.*

Exercise 3 *Let f be a convex function. Show that the subdifferential set $\partial f(x)$ is closed and convex for any x.*

Exercise 4 *Let f be a convex function.*

(a) Let $x \in \operatorname{dom} f$ and let $d \in \mathcal{R}^n$ ($d \neq 0$) be such that $x + \delta d \in \operatorname{dom} f$ for some $\delta > 0$. Consider the scalar function

$$\phi(\alpha) = \frac{f(x + \alpha d) - f(x)}{\alpha} \qquad \text{for } 0 < \alpha \leq \delta.$$

Using convexity of f show that $\phi(\alpha)$ is nondecreasing function of α.

(b) Suppose that $\partial f(x)$ is nonempty. Show the following relation

$$f'(x; d) \geq s^T d \qquad \text{for all } s \in \partial f(x).$$

Exercise 5 *Prove Theorem 3 of Lecture 19.*

Exercise 6 *Consider the sets C_1 and C_2 as defined in the proof of Theorem 4 of Lecture 18. Show that these sets are nonempty, convex, and disjoint.*