Homework 6: Gradient Methods

Exercise 1 Consider the scalar function $f(x) = \ln x$ for $x > 0$. Is this function strictly convex? Is it strongly convex?

Exercise 2 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a strongly convex function with constant $\nu > 0$.

(a) Show that every level set

$$L_f(\gamma) = \{x \in \mathbb{R}^n \mid f(x) \leq \gamma\}$$

is bounded.

(b) Consider minimization problem $\min_{x \in X} f(x)$ where $X \subseteq \mathbb{R}^n$ is closed and convex. Let $x^*$ be an optimal solution. Show that

$$\frac{\nu}{2} \|x^* - x\| \leq f(x) - f(x^*) \quad \text{for all } x \in X.$$

Exercise 3 Prove Theorem 1 of Lecture 13 (discussing convergence of the gradient projection method when the gradients are bounded).

Exercise 4 (Optimality Condition as Fixed Point) Consider convex minimization problem

$$\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in X,
\end{align*}$$

with convex $f : \mathbb{R}^n \to \mathbb{R}$ and convex closed set $X \subseteq \mathbb{R}^n$. Show that a vector $x^*$ is optimal if and only if

$$x^* = P_X[x^* - \tau \nabla f(x^*)] \quad \text{for any } \tau > 0.$$

Exercise 5 Prove Theorem 3 of Lecture 13 (discussing convergence rate of the gradient projection method when the gradients are Lipschitz).

Exercise 6 Show that the Newton decrement $\lambda(x) = \sqrt{\nabla f(x)^T [\nabla^2 f(x)]^{-1} \nabla f(x)}$ is affine invariant: that is show that $\lambda^2(x) = \lambda^2(Ax + b)$ for any $n \times n$ nonsingular matrix $A$ and $b \in \mathbb{R}^n$. 