Homework 6: Gradient Methods

Exercise 1 *Consider the scalar function $f(x) = \ln x$ for $x > 0$. Is this function strictly convex? Is it strongly convex?*

Exercise 2 *Let $f : \mathcal{R}^n \to \mathcal{R}$ be a strongly convex function with constant $\nu > 0$.*

(a) Show that every level set

$$L_f(\gamma) = \{x \in \mathcal{R}^n \mid f(x) \le \gamma\}$$

is bounded.

(b) Consider minimization problem $\min_{x \in X} f(x)$ where $X \subseteq \mathcal{R}^n$ is closed and convex. Let x^* be an optimal solution. Show that

$$\frac{\nu}{2} \|x^* - x\| \le f(x) - f(x^*) \qquad \text{for all } x \in X.$$

Exercise 3 *Prove Theorem 1 of Lecture 13 (discussing convergence of the gradient projection method when the gradients are bounded).*

Exercise 4 *(***Optimality Condition as Fixed Point***) Consider convex minimization problem*

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in X, \end{array}$$

with convex $f : \mathcal{R}^n \to \mathcal{R}$ and convex closed set $X \subseteq \mathcal{R}^n$. Show that a vector x^* is optimal if and only if

$$x^* = P_X[x^* - \tau \nabla f(x^*)] \qquad \text{for any } \tau > 0.$$

Exercise 5 *Prove Theorem 3 of Lecture 13 (discussing convergence rate of the gradient projection method when the gradients are Lipschitz).*

Exercise 6 *Show that the Newton decrement $\lambda(x) = \sqrt{\nabla f(x)^T [\nabla^2 f(x)]^{-1} \nabla f(x)}$ is affine invariant: that is show that $\lambda^2(x) = \lambda^2(Ax + b)$ for any $n \times n$ nonsingular matrix A and $b \in \mathcal{R}^n$.*