Homework 5: Duality, KKT and Sensitivity

Exercise 1 (Piecewise Linear Minimization) *Consider the problem*

$$\begin{array}{ll} \textit{minimize} & \max_{1\leq i\leq m}\{a_i^T x + b_i\} \\ \textit{subject to} & x \in \Re^n. \end{array} \tag{1}$$

(a) *Introducing new variables* $y_i = a_i^T x + b_i$*, formulate the problem in an equivalent form, and derive its dual problem.*

(b) *Formulate the original problem in (1) as a linear programming problem and derive its dual. Relate this LP dual to the dual obtained in part (a).*

Exercise 2 *Consider the problem*

$$\begin{array}{ll} \textit{minimize} & e^{-x} \\ \textit{subject to} & \frac{x^2}{y} \leq 0, \ (x,y) \in \mathcal{D}, \end{array}$$

with scalar variables x *and* y*, and* $\mathcal{D} = \{(x,y) \mid y > 0\}$.

(a) *Verify that this is a convex problem, and find its optimal value* f^*.

(b) *Does Slater condition hold for this problem?*

(c) *Formulate its dual, and find the dual optimal value* q^* *and the dual optimal solution* μ^**. What is the value of* $f^* - q^*$*?*

(d) *As a function of* u*, what is the optimal value* $p(u)$ *of the perturbed problem*

$$\begin{array}{ll} \textit{minimize} & e^{-x} \\ \textit{subject to} & \frac{x^2}{y} \leq u, \ (x,y) \in \mathcal{D}. \end{array}$$

Show that the global sensitivity relation

$$p(u) \geq p(0) - \mu^* u,$$

fails to hold.

Exercise 3 *Consider the equality constrained least-squares problem*

$$\begin{array}{ll} \textit{minimize} & \|Ax - b\|^2 \\ \textit{subject to} & Gx = h, \end{array}$$

where $A \in \Re^{m\times n}$ *with* $\mathrm{rank} A = n$*, and* $G \in \Re^{p\times n}$ *with* $\mathrm{rank} G = p$*. Give the KKT conditions, and derive the expressions for the primal optimal solution and the dual optimal solution.*

Exercise 4 *Consider the following LP*

$$\begin{array}{ll} \textit{minimize} & c^T x \\ \textit{subject to} & Ax \geq c, \\ & x \geq 0, \end{array}$$

where $A \in \Re^{n\times n}$ *is a symmetric matrix. Let* x^* *satisfy* $Ax^* = c$ *and* $x^* \geq 0$*. Show that* x^* *is an optimal solution of the LP.*

Exercise 5 *Consider the problem*

$$\begin{array}{ll} \textit{minimize} & x_1 + x_2 \\ \textit{subject to} & x_1 + 2x_2 = \theta, \\ & x_1 \geq 0, \ x_2 \geq 0, \end{array}$$

where $x_1, x_2 \in \Re$.

(a) *Find the optimal value* $p(\theta)$ *as a function of* θ.

(b) *Also, find an optimal solution* $x^*(\theta)$ *as a function of* θ.

(c) *Determine the set* $X^*(\theta)$ *of the optimal solutions as a function* θ.

Exercise 6 *Consider the LP in the standard form*

$$\begin{array}{ll} \textit{minimize} & c^T x \\ \textit{subject to} & Ax = b, \\ & x \geq 0. \end{array}$$

Let x^* *and* p^* *be respectively a primal optimal and a dual optimal for this problem.*

(a) *Let* $\bar{x}$ *be an optimal solution of the LP when* c *is replaced by some* $\bar{c}$*. Show that*

$$(\bar{c} - c)^T(\bar{x} - x^*) \leq 0.$$

(b) *Let the cost be fixed at* c*. Let* $\tilde{x}$ *be an optimal solution when* b *is replaced with* $\tilde{b}$*. Show that*

$$(p^*)^T(\tilde{b} - b) \leq c^T(\tilde{x} - x^*).$$