Homework 4: Lagrangian Duality

Exercise 1 *Consider a general optimization problem of the form*

$$\begin{array}{ll} \textit{minimize} & f(x) \\ \textit{subject to} & g_j(x) \le 0,\ j = 1, \ldots, m, \\ & h_i(x) = 0,\ i = 1, \ldots, p, \\ & x \in X, \end{array}$$

where $X \subseteq \Re^n$, $f : \Re^n \to \Re$, *each* $g_j : \Re^n \to \Re$, *and each* $h_i : \Re^n \to \Re$. *Assume that the optimal value* f^* *of the problem is finite. Consider the dual problem obtained by assigning a Lagrange multiplier* $\mu_j \ge 0$ *for each constraint* $g_j(x) \le 0$ *and a Lagrange multiplier* $\lambda_i \ge 0$ *for each constraint* $h_i(x) = 0$. *Show the lower bounding property*

$$q(\mu, \lambda) \le f^*,$$

where q *is the dual function,* $\mu = [\mu_1, \ldots, \mu_m]^T$ *and* $\lambda = [\lambda_1, \ldots \lambda_p]^T$.

Exercise 2 *Consider an LP in the standard form*

$$\begin{array}{ll} \textit{minimize} & c^T x \\ \textit{subject to} & Ax = b, \\ & x \ge 0, \end{array}$$

and its corresponding dual

$$\begin{array}{ll} \textit{maximize} & b^T p \\ \textit{subject to} & A^T p \le c. \end{array}$$

where $x \in \Re^n$, $b \in \Re^m$, *and* $A \in \Re^m \times \Re^n$. *Transform the dual in a minimization problem and find its dual. Show that this dual (of the dual) is the same as the original LP.*

Exercise 3 *Consider the following optimization problem in* $\Re^2$*:*

$$\begin{array}{ll} \textit{minimize} & e^{-x_2} \\ \textit{subject to} & \|x\| \le x_1,\ x_2 \ge 0. \end{array}$$

(a) *What is the constraint set? What is the optimal value of the problem?*

(b) *Consider a dual problem obtained by assigning a Lagrange multiplier only to the constraint* $x_2 \ge 0$. *What is the dual optimal value? Is there a duality gap?*

(c) *Consider a dual problem obtained by assigning a Lagrange multiplier only to the constraint* $\|x\| \le x_1$. *What is the dual optimal value in this case? Is there a duality gap?*

(d) *What is your interpretation of the meaning of the results obtained in (b) and (c)?*

Exercise 4 *Prove the following* **Relax-all rule***: Given a minimization problem*

$$\begin{array}{ll} \textit{minimize} & f(x) \\ \textit{subject to} & g_j(x) \leq 0, \ j = 1, \ldots, m, \\ & x \in \Re^n, \end{array}$$

where $X \subseteq \Re^n$, $f : \Re^n \to \Re$, *and each* $g_j : \Re^n \to \Re$. *Assume that the optimal value* f^* *of the problem is finite.*

Consider a dual problem obtained by assigning a Lagrange multiplier $\mu_j > 0$ *to each constraint* $g_j(x) \leq 0$, *and assume that there is no duality gap, i.e.,* $q^* = f^*$. *Then, there is no duality gap when considering any other dual problem obtained by assigning Lagrange multipliers* $\mu_j > 0$ *to constraints* $g_j(x) \leq 0$ *with* $j \in J$ *for a subset* $J \subset \{1, \ldots, m\}$. *Specifically, for any* $J \subset \{1, \ldots, m\}$, *the partial dual problem*

$$\begin{array}{ll} \textit{maximize} & \tilde{q}(\tilde{\mu}) \\ \textit{subject to} & \tilde{\mu}_j(x) \geq 0, \ j \in J \end{array} \tag{1}$$

with

$$q(\tilde{\mu}) = \inf_{g_j(x) \leq 0, \ j \notin J} \left\{ f(x) + \sum_{j \in J} \tilde{\mu}_j g_j(x) \right\},$$

we have $\tilde{q}^* = f^*$, *where* $\tilde{q}^*$ *is the optimal value of the partial dual problem in (1).*

Exercise 5 *(*Analytic centering*) Consider the following problem*

$$\begin{array}{ll} \textit{minimize} & -\sum_{i=1}^m \ln(b_i - a_i^T x) \\ \textit{subject to} & x \in \Re^n, \end{array} \tag{2}$$

with domain $\{x \in \Re^n \mid a_i^T x < b_i, \ i = 1, \ldots, m\}$. *Derive its dual problem by first introducing new variables* y_i *and then introducing equality constraints* $y_i = b_i - a_i^T x$ *for* $i = 1, \ldots, m$.

Remark*: The solution to the problem in (2) is known as* **analytic center** *of the linear inequalities* $a_i^t x \leq b_i$, $i = 1, \ldots, m$. *The function* $-\ln(b_i - a_i^T x)$ *is referred to as a* **log-barrier** *function. This function plays an important role in the interior point method, which we study later on.*

Exercise 6 *Consider the following problem*

$$\begin{array}{ll} \textit{minimize} & f(x) \\ \textit{subject to} & g_j(x) \leq 0, \ j = 1, \ldots, m, \end{array} \tag{3}$$

where the function $f : \Re^n \to \Re$ *and each* $g_j : \Re^n \to \Re$ *are differentiable and convex. Let* $h_j : \Re \to \Re$ *be increasing differentiable convex functions. Show that*

$$\phi(x) = f(x) + \sum_{j=1}^m h_j(g_j(x))$$

is convex.

Suppose that $\tilde{x}$ *minimizes* $\phi(x)$ *over* $\Re^n$. *Show how to use* $\tilde{x}$ *to find a feasible vector for the dual of the problem in (3).*