Homework 3: Projection Theorem and Separating Hyperplanes

Exercise 1 *Provide an example of a nonconvex set C for which the distance function $dist(\cdot, C)$ is not convex.*

Exercise 2 *Strictly Separating Hyperplane Theorem states that given two nonempty disjoint convex sets C and D such that $C - D$ is closed, the sets C and D can be strictly separated, i.e., there exists a vector $a \neq 0$ such that*

$$\sup_{x \in C} a^T x < \inf_{z \in D} a^T z.$$

Prove the Strictly Separating Hyperplane Theorem. Hint: Argue that $0 \notin C - D$ and show that 0 can be strictly separated from $C - D$.

Exercise 3 *Give an example of two nonempty closed convex disjoint sets that cannot be strictly separated.*

Exercise 4 *(Extreme Points of a Convex Set)*
A vector $\hat{x}$ in a convex set C is said to be an extreme point of C when $\hat{x}$ cannot be expressed as a nontrivial convex combination of any other vectors in C, i.e., there are NO vectors $x_1 \in C$ and $x_2 \in C$, $x_1 \neq x_2$, such that

$$\hat{x} = \alpha x_1 + (1 - \alpha) x_2 \qquad \textit{for some } \alpha \in (0, 1).$$

For example, when C is a square, each corner of C is an extreme point.

Show the following property of extreme points:

Let C be a convex set, and H be a supporting hyperplane of C such that the set intersection $T = C \cap H$ is nonempty. Then, every extreme point of T is an extreme point of C.