Homework 3: Projection Theorem and Separating Hyperplanes

Exercise 1 Provide an example of a nonconvex set $C$ for which the distance function $\text{dist}(\cdot, C)$ is not convex.

Exercise 2 Strictly Separating Hyperplane Theorem states that given two nonempty disjoint convex sets $C$ and $D$ such that $C - D$ is closed, the sets $C$ and $D$ can be strictly separated, i.e., there exists a vector $a \neq 0$ such that

$$\sup_{x \in C} a^T x < \inf_{z \in D} a^T z.$$ 

Prove the Strictly Separating Hyperplane Theorem. Hint: Argue that $0 \notin C - D$ and show that $0$ can be strictly separated from $C - D$.

Exercise 3 Give an example of two nonempty closed convex disjoint sets that cannot be strictly separated.

Exercise 4 (Extreme Points of a Convex Set)
A vector $\hat{x}$ in a convex set $C$ is said to be an extreme point of $C$ when $\hat{x}$ cannot be expressed as a nontrivial convex combination of any other vectors in $C$, i.e., there are NO vectors $x_1 \in C$ and $x_2 \in C$, $x_1 \neq x_2$, such that

$$\hat{x} = \alpha x_1 + (1 - \alpha) x_2 \quad \text{for some} \ \alpha \in (0,1).$$

For example, when $C$ is a square, each corner of $C$ is an extreme point.

Show the following property of extreme points:

Let $C$ be a convex set, and $H$ be a supporting hyperplane of $C$ such that the set intersection $T = C \cap H$ is nonempty. Then, every extreme point of $T$ is an extreme point of $C$. 