Homework 2: Convex Functions and Optimality Conditions

Exercise 1 *Find the domain and find out whether the function is convex or not for each of the following functions:*

(a) $f(x) = \|x\|^4$ *with* $x \in \mathcal{R}^n$.

(b) $f(x) = \sum_{i=1}^{n} x_i \ln x_i$ *with each* $x_i \in \mathcal{R}$ *and* $x_i > 0$.

(c) $f(x) = \sup_{\mu \in \mathcal{R}} \{(x_1 + x_2)\mu\} + x_1 x_2$.

(d) $f(x) = \sup_{\mu \in \mathcal{R}^2_+} \{(x_1+x_2-1)\mu_1+(x_1-x_2-2)\mu_2\}+x_1^2+x_2^2$, *where* R^2_+ *is nonnegative orthant in* $\mathcal{R}^2$.

Exercise 2 *Determine whether the following sets are convex or not.*

(a) *The ellipsoid given by*

$$\mathcal{E} = \{x \in \mathcal{R}^n \mid (x - \hat{x})^T P(x - \hat{x}) \leq 1\},$$

where $\hat{x}$ *is the center of the ellipsoid and* P *is a positive definite matrix.*

(b) *The set* C *given by*

$$C = \{(x_1, x_2) \in \mathcal{R}^2 \mid x_1 x_2 \leq 1,\ x_1 < 0\}.$$

(c) *The set* C *given by*

$$C = \{x \in \mathcal{R}^2 \mid x_1^{\alpha_1} x_2^{\alpha_2} \geq a,\ x_1 > 0,\ x_2 > 0\},$$

where $\alpha_1 > 0$, $\alpha_2 > 0$, *and* $a > 0$ *are some scalars.*

Exercise 3 *Consider the following optimization problem*

$$\begin{array}{ll} \textit{minimize} & x_1^2 + (x_2 + 1)^2 \\ \textit{subject to} & -1 \leq x_1 \leq 1,\ x_2 \geq 0 \end{array}$$

Use the optimality condition to show that the vector $(0, 0)$ *is a unique optimal solution.*

Exercise 4 *Consider the following unconstrained optimization problem*

$$\begin{array}{ll} \textit{minimize} & \sum_{j=1}^{m} \|x - v_j\|^2 \\ \textit{subject to} & x \in \mathcal{R}^n, \end{array}$$

where $v_1, \ldots, v_m$ *are some given vectors in* $\mathcal{R}^n$. *Use the optimality condition to find an optimal solution to the problem. Is the solution unique?*

Exercise 5 *Consider the following constrained optimization problem*

$$\begin{array}{ll} \textit{minimize} & f(x) \\ \textit{subject to} & x \in X, \end{array}$$

where f is a convex and continuously differentiable function, and $X \subseteq \mathcal{R}^n$ is a box constraint of the form

$$X = \{x \in \mathcal{R}^n \mid a_i \leq x_i \leq b_i \textit{ for all } i\},$$

for some scalars a_i and b_i. Using the optimality conditions verify that x^* is an optimal solution if and only if x^* satisfies the following relations for all $i = 1, \ldots, n$:

$$\frac{\partial f(x^*)}{\partial x_i} \geq 0 \quad \textit{if} \quad x_i^* = a_i,$$

$$\frac{\partial f(x^*)}{\partial x_i} = 0 \quad \textit{if} \quad a_i < x_i^* < b_i,$$

$$\frac{\partial f(x^*)}{\partial x_i} \leq 0 \quad \textit{if} \quad x_i^* = b_i.$$

Exercise 6 *Consider the following constrained optimization problem*

$$\begin{array}{ll} \textit{minimize} & \|x\|^2 \\ \textit{subject to} & a^T x = b, \end{array}$$

where $a \in \mathcal{R}^n$ with $a \neq 0$ and $b \in \mathcal{R}$. Using the optimality conditions, find and optimal solution to the problem. Is the solution unique? What is the optimal value?