Homework 2: Convex Functions and Optimality Conditions

Exercise 1  Find the domain and find out whether the function is convex or not for each of the following functions:

(a) \( f(x) = \|x\|^4 \) with \( x \in \mathbb{R}^n \).
(b) \( f(x) = \sum_{i=1}^{n} x_i \ln x_i \) with each \( x_i \in \mathbb{R} \) and \( x_i > 0 \).
(c) \( f(x) = \sup_{\mu \in \mathbb{R}} \{(x_1 + x_2)\mu\} + x_1 x_2 \).
(d) \( f(x) = \sup_{\mu \in \mathbb{R}^2_+} \{(x_1 + x_2 - 1)\mu_1 + (x_1 - x_2 - 2)\mu_2\} + x_1^2 + x_2^2 \), where \( \mathbb{R}^2_+ \) is nonnegative orthant in \( \mathbb{R}^2 \).

Exercise 2  Determine whether the following sets are convex or not.

(a) The ellipsoid given by
\[
\mathcal{E} = \{x \in \mathbb{R}^n \mid (x - \hat{x})^T P(x - \hat{x}) \leq 1\},
\]
where \( \hat{x} \) is the center of the ellipsoid and \( P \) is a positive definite matrix.
(b) The set \( C \) given by
\[
C = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 x_2 \leq 1, \ x_1 < 0\}.
\]
(c) The set \( C \) given by
\[
C = \{x \in \mathbb{R}^2 \mid x_1^{\alpha_1} x_2^{\alpha_2} \geq a, \ x_1 > 0, \ x_2 > 0\},
\]
where \( \alpha_1 > 0, \alpha_2 > 0, \) and \( a > 0 \) are some scalars.

Exercise 3  Consider the following optimization problem
\[
\begin{align*}
\text{minimize} & \quad x_1^2 + (x_2 + 1)^2 \\
\text{subject to} & \quad -1 \leq x_1 \leq 1, \ x_2 \geq 0
\end{align*}
\]
Use the optimality condition to show that the vector \((0, 0)\) is a unique optimal solution.

Exercise 4  Consider the following unconstrained optimization problem
\[
\begin{align*}
\text{minimize} & \quad \sum_{j=1}^{m} ||x - v_j||^2 \\
\text{subject to} & \quad x \in \mathbb{R}^n,
\end{align*}
\]
where \( v_1, \ldots, v_m \) are some given vectors in \( \mathbb{R}^n \). Use the optimality condition to find an optimal solution to the problem. Is the solution unique?
Exercise 5 Consider the following constrained optimization problem

\[
\begin{align*}
\text{minimize} \quad & f(x) \\
\text{subject to} \quad & x \in X,
\end{align*}
\]

where \( f \) is a convex and continuously differentiable function, and \( X \subseteq \mathbb{R}^n \) is a box constraint of the form

\[
X = \{ x \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i \text{ for all } i \},
\]

for some scalars \( a_i \) and \( b_i \). Using the optimality conditions verify that \( x^* \) is an optimal solution if and only if \( x^* \) satisfies the following relations for all \( i = 1, \ldots, n \):

\[
\begin{align*}
\frac{\partial f(x^*)}{\partial x_i} & \geq 0 \quad \text{if} \quad x_i^* = a_i, \\
\frac{\partial f(x^*)}{\partial x_i} & = 0 \quad \text{if} \quad a_i < x_i^* < b_i, \\
\frac{\partial f(x^*)}{\partial x_i} & \leq 0 \quad \text{if} \quad x_i^* = b_i.
\end{align*}
\]

Exercise 6 Consider the following constrained optimization problem

\[
\begin{align*}
\text{minimize} \quad & \|x\|^2 \\
\text{subject to} \quad & a^T x = b,
\end{align*}
\]

where \( a \in \mathbb{R}^n \) with \( a \neq 0 \) and \( b \in \mathbb{R} \). Using the optimality conditions, find and optimal solution to the problem. Is the solution unique? What is the optimal value?